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Structural Optimization

Lecture Notes

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Értékközpontúság, nyitottság és befogadó szemlélet
egy 21. századi felsőoktatási modellben

STRUCTURAL OPTIMIZATION

LECTURE NOTES

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PREFACE

The aim to write of this teaching material is to provide an outline about the essence of the structural engineering problems. However the author of this lecture notes is a researcher of special parts of the structural optimization the main aim was here to give a tool for students, a tool to offer solution techniques, to give a short description of the theoretical backgrounds of structural optimization problems.

The scope of this booklet is the mathematical formulation perfectly elastic and elastic plastic structural optimization problems. According to the complexity of usual structural optimization problems, throughout simple examples we present the solution methodology as well using the program package of *Wolfram Mathematica*[1] The introduction contents the basic principles of the structural optimization modelling and the classification in computational methodology point of view. Second chapter related to linear elastic structural optimization problems where the objective function is the weight or volume of the whole structure subject to several structural constraints such as stress, displacements, local and global structural instability. The effects of the constraints are demonstrated by graphical interpretation of the results for 2D problems. Third chapter presents elasto-plastic and perfectly plastic optimization formulations. In this chapter different collapse states and two examples are considered to emphasize the main advantages of plastic design over the traditional elastic design.

Fourth chapter deals with the most simple solution technique, namely the linear programming method because this is the basics of the nonlinear programming methods. Throughout simple 2D and 3D examples the simplex method are shown manually and compared the results using *Wolfram Mathematica.*

Fifth chapter offers a higher order mechanical modelling, where large deflection behavior is supposed. The structural control and stability criteria are computed based on the principle of minimum potential energy function. This chapter deals with the theoretical background and throughout two examples the geometrically nonlinear structural modelling is presented. Numerical computational code is given step by step in order to demonstrate the already highly nonlinear and nonconvex problem complexity.

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[1] https://www.wolfram.com/mathematica/online/

The topic of sixth chapter points beyond the median level of presentations for graduate students. However, this part of the structural optimization requires already higher level of knowledge of numerical mathematics, the author provides for graduate students those are interesting on a scientific research of special problems in structural optimization or apply to be doctoral student. Some texts are published in scientific journals by the author to present the current state of the field.

The author would like to express her acknowledgement for the financial support of the Project No. EFOP - 3.4.3-16-2016-00005 „Felsőoktatási intézményi fejlesztések a felsőfokú oktatás minőségének és hozzáférhetőségének együttes javítása érdekében”.

1.INTRODUCTION

1.1 Modelling of Structural Optimization Problems

The traditional structural design is an iterative procedure supposing an initial design based on empirical formulas of structural engineering process. Starting from a pre-conceptual structure where the applied materials, geometry, and the cross sections are given, the internal forces are computed. In the following step the designer compare the obtained stresses, strains, or displacements with the capacity of the selected structural materials and the given structural geometry. If this structure satisfies the main requirements the process terminates.

In order to obtain a better design the structural optimization let us to define several free parameters that will be hereinafter the so called *design variables* during the structural design process. The goal of the optimal design might be to obtain a simple light-weight structure or a more reliable structure. However generally we are able to vindicate more complex requirements. The constructor could be prescribe different concepts as the goals of the structural design as multiple *goal functions* or in other name the *objectives* of the optimal design.

In order to avoid any type of the structural collapse the structural requirements are considered during the searching process while we are looking for the minimal weight, minimal volume, or minimal cost structure. We have to satisfy the stress, strain, displacement, stability and any others criteria such that in normal rule of the traditional structural design. Therefore, we will apply the main relationships of static equilibrium, compatibility criteria, and material law, in linear or nonlinear cases depending on the structural ability. The structural rules will be considered as *equality or inequality constraints* of the structural optimization.

The essence of the structural optimization is described above verbally, but in order to solve the problem we need to define the mathematical formulas of the optimization problem. The objective functions are minimization where two types of variables are supposed, design variables and state variables.

The objective function:

$$min\,\{f_1(X,Y), f_2(X,Y), \dots, f_n(X,Y)\} \tag{1.1}$$

subject to:

$$g_j(X,Y) = 0,\ j = 1,2,\dots,k \tag{1.2}$$

$$h_j(X,Y) \leq 0,\ j = 1,2,\dots,l \tag{1.3}$$

where $g_j(X,Y)$ are the equality constraints, $h_j(X,Y)$ are the inequality constraints, X is the vector of the design variables, and Y is the vector of the state variables.

In the mathematical formulas (1.1-1.3) described above very likely either of them will be nonlinear. Moreover, most structural engineering optimization problems are highly nonlinear and non-convex. Consequently, the applied solution techniques must be nonlinear. The problem is typically large, and the evaluation of the functions and gradients is expensive due to their implicit dependence on design variables (see in (Csébfalvi, 2009).

The traditional engineering optimization algorithms are based on nonlinear programming methods that require substantial gradient information and usually seek to improve the solution in the neighborhood of a starting point. Many real-world engineering optimization problems, however, are very complex in nature and quite difficult to solve using these algorithms. If there is more than one local optimum in the problem, the results may depend on the selection of an initial point, and the obtained optimal solution may not necessarily be the global optimum.

The computational drawbacks of existing numerical methods have forced researchers to rely on metaheuristic algorithms based on simulations to solve engineering optimization problems. The common factor in metaheuristic algorithms is that they combine rules and randomness to imitate natural phenomena. Csébfalvi (Csébfalvi, 2009) describes a hybrid metaheuristic for engineering optimization problems with continuous design variables.

The highlights of this work are to present the complexity and computational difficulties of structural optimization problems. However, the subject doesn’t cover the knowledge of the full mathematical background. Therefore, in order to solve the thematic examples, the Author proposes the usage of the program package of *Wolfram Mathematica* which is available for students in the computer laboratories of the Faculty of Engineering and Information Technology of University of Pécs. Further instructions for the application of the *Wolfram Mathematica* will be described in sub-sequential chapter connection with relevant subjects.

1.2 Classification of Structural Optimization Problems

There are many types of classifications to distinguish the optimization methods and techniques. One of them might be the nature of objective functions, or the specification of the structural constraints and the type of the structures. Christensen and Klarbring (Christensen, 2009) proposed a specification system where basically three different types of structural optimizations are distinguished:

Sizing optimization: This is when the objective function and constraints are given in terms of X where of X is some type of structural thickness, i.e., cross-sectional areas of truss members, or the thickness distribution of a sheet. A sizing optimization problem for a truss structure is shown in Fig. 1.1.

Shape optimization: In this case of Y represents the form or contour of some part of the boundary of the structural domain. Think of a solid body, the state of which is described by a set of partial differential equations. The optimization consists in choosing the integration domain for the differential equations in an optimal way. Note that the connectivity of the structure is not changed by shape optimization: new boundaries are not formed. A three-dimensional shape optimization problem is seen in Fig. 1.2.

Topology optimization: This is the most general form of structural optimization, where the design variables might be complex, e.g. cross-sectional areas and geometrical variables are given in terms of X, Y. In a discrete case, such as for a truss, it is achieved by taking cross-sectional areas of truss members as design variables, and then allowing these variables to take the value zero, i.e., bars are removed from the truss. In this way the connectivity of nodes is variable so we may say that the topology of the truss changes, see Fig. 1.3.

If instead of a discrete structure we think of a continuum-type structure such as a two-dimensional sheet, then topology changes can be achieved by letting the thickness of the sheet take the value zero. If pure topological features are optimized, the optimal thickness should take only two values: 0 and a fixed maximum sheet thickness.

Naturally, the types of structural optimizations listing above could be shared into further sub-types concerning different design variables. The other point of view to distinguish the optimization types, the selection of different solution techniques (e.g. linear, or nonlinear optimization methods).

Finally we could select different types of structural modelling, e.g. trusses, slabs, beams, or shall structures.

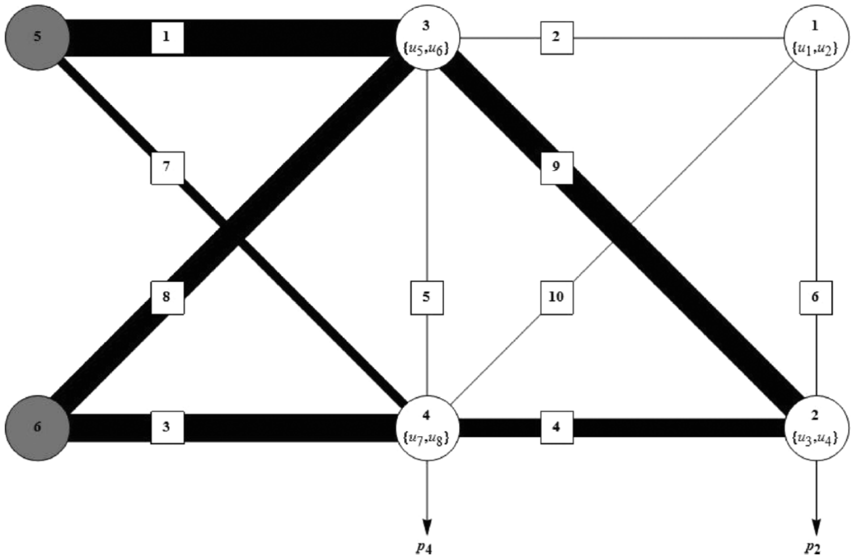


Figure 1.1 Nominal solution of minimal weight *sizing optimization.*[2]

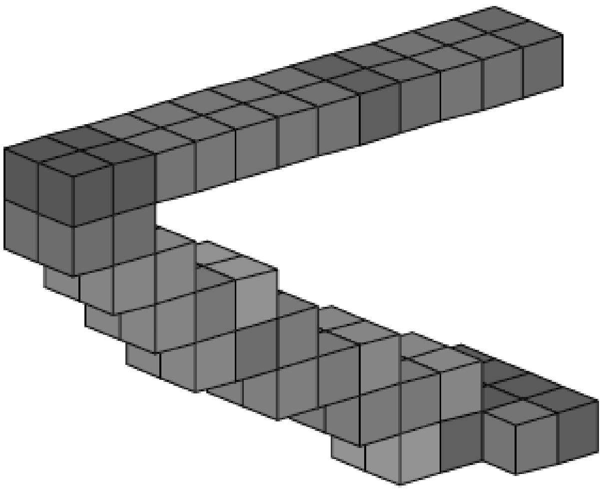


Figure 1.2 *Shaping optimization of a 3D cantilever beam*[3]

[2] Csébfalvi A., A new theoretical approach for robust truss optimization

[3] Csébfalvi A, 3D Benchmark Results for Robust Structural Optimization……

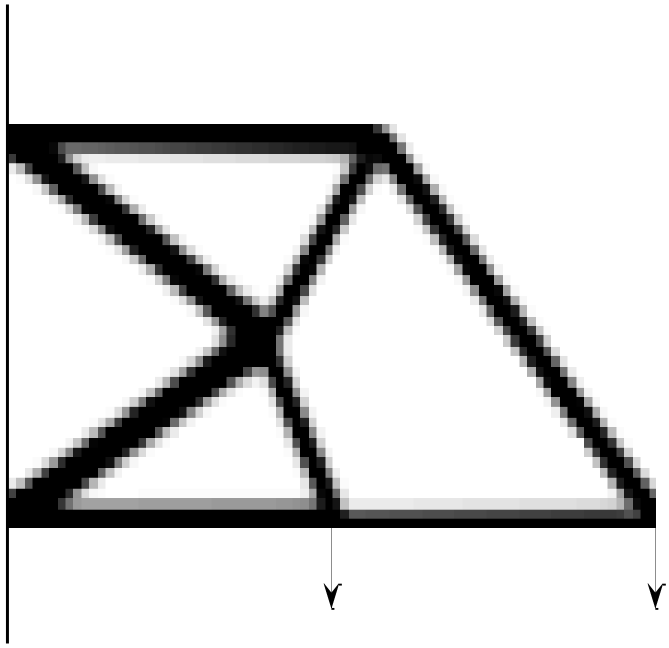


Figure 1.3 Nominal-compliance-minimal shape of topology optimization[4]

Approaching from the properties of the applied material, the modelling will be basically two types:

1. Idealized *linear elastic*, where the optimization process generally requires before a linear or non-linear structural analysis depending on the structural behavior,
2. Idealized *linear elastic – perfectly plastic*, where the structural modelling doesn't requires the structural analysis simultaneously, and led to a simpler optimization process.

We have to note that "*The plastic analysis cannot replace the elastic analysis but supplements it by giving useful information about the collapse load and the mode of the collapse*"[5]. In some special cases, where large deformations, or structural instabilities (see e.g. shallow space structures in Fig. 1.4-1.5) are considered the plastic analysis seems very useful to give an estimated solution of optimal structural design. However, the elastic instability analysis requires the evaluation of large scale problems inside of a heuristic optimization technique.

[4] Csébfalvi A, Lógó J, A critical analysis of expected-compliance…

[5] Kirsch (1981) *Optimum Structural Design, Concepts, Methods and Applications*

According to the limitation of pages of this book the consideration of structural optimization will be reduced to only to rod structures i.e. optimization problems related to trusses and beam structures are presented in the consecutive chapters.

2. OPTIMIZATION OF ELASTIC STUCTURES

Considering a statically determinate structure, the most frequently occurring examples are the minimal weight design or minimal volume design subject to structural equilibrium criteria as equilibrium constraints. In order to satisfy the further structural requirements, stress, displacement, and buckling constraints are imposed as inequality constraints of the structural optimization process.

Considering a statically indeterminate structure, the formulation of objective function is same as defined above, but in case of computation of structural constraints we need to satisfy not only the static equilibrium equation systems but more over we need to satisfy the compatibility and constitution constraints as well.

The mathematical formulation of the structural optimization problem is the following:

$$min\, f(A_i) = \rho \sum A_i L_i, i = 1,2, \dots, m \tag{2.1}$$

$$g_j(A_i) = 0,\ j = 1,2, \dots, k \tag{2.2}$$

$$h_j(A_j) \leq 0,\ j = 1,2, \dots, l \tag{2.3}$$

where $g_j(A_i)$ are the equality constraints, $h_j(A_i)$ are the inequality constraints in terms of cross sections, A_i is the vector of the cross section design variables, L_i is the vector of the length of elements, and ρ is the density of the applied material.

Using the optimization problem described above in case of large structures or in case when we apply large number of design variables we have to simplify the computation of structural constraints.

In the following subchapters the computational background of structural constraints are presented. Subsequently, through simple examples statically determinate and statically indeterminate structures are considered to demonstrate the problem formulation, mathematical modelling, and its solution techniques using the *Wolfram Mathematica* program package.

2.1 Structural Constraints

In order to satisfy the equality and inequality constraints defined above in formula (2.2) and (2.3) we have to know the exact mechanical modelling, the structural behavior of an optimization problem. This chapter presents an overview about the basic knowledge of structural analysis. Refer to Aslam Kassimali, in book of Matrix Analysis of Structures (Kassimali, 2012) he stated, "Structural analysis, which is an integral part of any structural engineering project, is the process of predicting the performance of a given structure under a prescribed loading condition. The performance characteristics usually of interest in structural design are: (a) stresses or stress resultants (i.e., axial forces, shears, and bending moments); (b) deflections; and (c) support reactions. Thus, the analysis of a structure typically involves the determination of these quantities as caused by the given loads and/or other external effects (such as support displacements and temperature changes)". Basically two types of structural optimizations are distinguished, the elastic and plastic optimizations in terms of the applied material law and modelling. In the traditional structural analysis we suppose a perfectly linear elastic material where the Hook's law is valid in the full range of loading.

In subsequent chapters of this book, we suppose that readers are familiar with classical methods of structural analysis and its extension to matrix method and finite element analysis. We present that both matrix and classical methods are based on the same fundamental principles—but that the fundamental relationships of equilibrium, compatibility, and member stiffness are now expressed in the form of matrix equations, so that the numerical computations can be efficiently performed on a computer. However, the finite element modelling is would be more complex in structural analysis, the final description of the small displacements based finite element method results the same formula of stiffness matrix as matrix method.

Structural analysis, in general, involves a triplet of *equilibrium equations (1)*, *compatibility conditions (2)*, and *constitutive relations (3)*, subsequently described by the following matrix equilibrium equation systems:

$$\mathbf{G}\,\mathbf{s} + \mathbf{q} = 0 \tag{2.4}$$

$$\mathbf{G}^T\mathbf{u} + \boldsymbol{\delta} + \mathbf{t} = 0 \tag{2.5}$$

$$\boldsymbol{\delta} = \mathbf{F}\,\mathbf{s} \tag{2.6}$$

where $\mathbf{G}$ is a geometry matrix which transforms the vector $\mathbf{s}$, the vector of element forces from local coordinate system into the global coordinate system, and $\mathbf{q}$ is the vector of external loads in formula (2.4) equilibrium equation system. The $\mathbf{G}^T$ matrix is the transpose of geometry matrix which transforms the vector **u**, the vector of nodal displacements from global coordinate system into the local coordinate system. The $\boldsymbol{\delta}$ is the vector of the change in length of the elements computed from the internal forces, and $\mathbf{t}$ is the vector of the change in length causes any other effects as internal forces, e. g. changes of the temperature or building imprecision in formula (2.5). The formula (2.6) is actually the Hook's law, where **F** is the flexibility matrix of the structure.

In case of *statically determinate structure*s and only stress constraints are imposed, the structural analysis reduced to *equilibrium equations* from the triplet of formulas (2.4-2.6). But, if both the stress and displacements are constrained we need the whole triplet requirements described above to solve the optimization problem.

In case of *statically indeterminate structure*s the *matrix direct method* or the *displacements based stiffness method* is proposed which is formulated in a *block matrix* below:

$$\begin{bmatrix} \mathbf{0} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{F} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} = 0. \tag{2.7}$$

The most simple solution of formula (2.7) the direct matrix method when the inverse matrix could be easily computed e. g. in case of a relatively small example.

In any other cases the *displacements based stiffness method* is applied, where the structural stiffness is computed from the geometrical matrix and flexibility matrix using the following formula obtained from the triplet of *equilibrium equations (1)*, *compatibility conditions (2),* and *constitutive relations (3):*

$$\mathbf{K}\,\mathbf{u} = \mathbf{q}, \tag{2.8}$$

where **K** is the structural stiffness matrix computed from the following formula:

$$\mathbf{K} = \mathbf{G}\,\mathbf{F}^{-1}\,\mathbf{G}^T. \tag{2.9}$$

The nodal displacements of the free joints are computed from (2.8) formula:

$$\mathbf{u} = \mathbf{K}^{-1}\,\mathbf{q}. \tag{2.10}$$

The nodal displacements replaced to compatibility equilibrium equation system, the member forces are obtained from the following formula:

$$\mathbf{s} = -\ \mathbf{F}^{-1}\,\mathbf{G}\,\mathbf{u}. \tag{2.11}$$

The inverse of the flexibility matrix is the structural stiffness in local coordinate system.

In subsequent chapter the constraints are determine using the basic formulas of structural analysis, where the linear elastic structural behavior is supposed. The further computation of the optimal structural design requires a deeper knowledge of the advanced structural analysis. There are several books[6] available for readers to refresh the already learned teaching materials or get a stronger cognitive skill on the field of structural design.

2.2 Minimal Weight Design Subject to Stress Constraints

Consider the statically determinate four-bar truss in Fig. 2.1. The geometry is given by the angle in between member 1 and 4, and member 3 and 4 ($\alpha = 30°$). Therefore, the lengths of the elements 1-3 is L, and the length of member 4 is $\sqrt{3}\,L$. The objective function of the optimization is the minimal weight subject to stress limit only. The design variables are the area of the cross sections A_1 *and* A_2 that have to satisfy the stress limit σ_0 and the positivity criteria (see in Fig. 2.2).

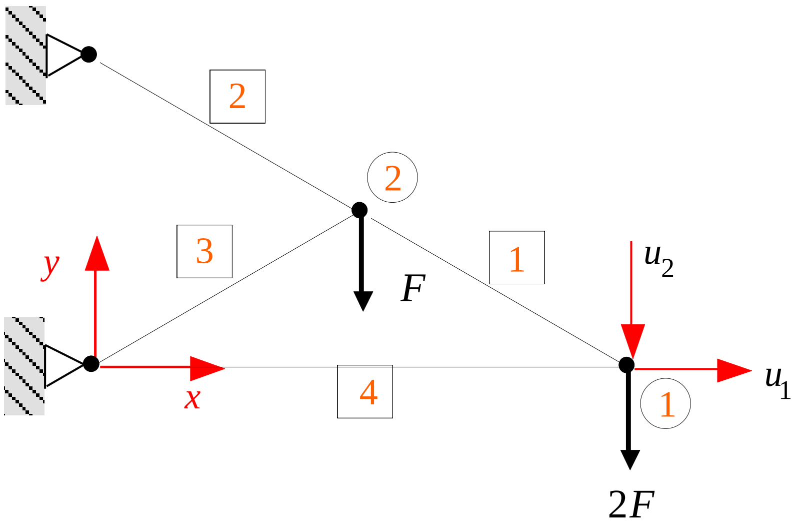


Figure 2.1 Topology of the four-bar truss (the free joints and truss members are numbering)

[6] Kassimali A. (2012) *Matrix Analysis of Structures*, SI Version, 2nd Edition ….

The mathematical formulation of the truss optimization problem in terms of design variables:

$$min\ f(A_1,\ A_2) = \rho L(3A_1 + \sqrt{3}\ A_2) \tag{2.12}$$

$$A_1 \geq 0;\ A_2 \geq 0 \tag{2.13}$$

$$\sigma_1 \leq \sigma_0\,;\sigma_2 \leq \sigma_0 \tag{2.14}$$

In order to determine the stress constraints in formula (2.14) we need to compute the internal forces of the truss elements. Consider the equilibrium equation criteria for nodal joints supposed tension forces in each element.

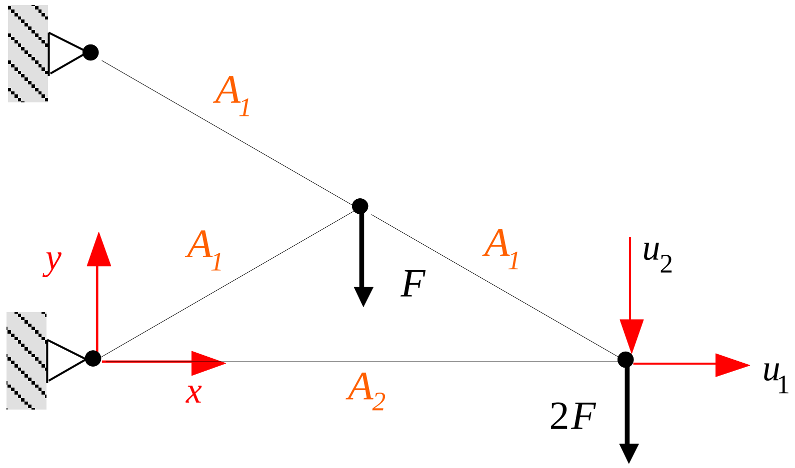


Figure 2.2 Design variables of the four-bar truss

The directions of components are given in x, y coordinate system (see Fig. 2.1).

$$\sum F_{ix} = 0, i = 1,2, \dots ,4 \tag{2.15}$$

$$\sum F_{iy} = 0, i = 1,2, \dots ,4 \tag{2.16}$$

Substituting the components of the internal forces into formula (2.1) and (2.2) for both joints 1 and 2:

$$-S_1 \cos \propto - S_4 = 0 \tag{2.17}$$

$$S_1 \sin \propto - 2F = 0 \tag{2.18}$$

$$S_1 \cos \propto - S_2 \cos \propto - S_3 \cos \propto = 0 \tag{2.19}$$

$$-S_1 \sin \propto + S_2 \sin \propto - S_3 \sin \propto - F = 0 \tag{2.20}$$

where S_i , $i = 1,2,\ldots,4$ are the internal forces of elements.

Replace to (2.4) matrix formula the equations (2.17-2.20), the geometrical matrix is described in terms of coefficients of member forces, and finally a matrix equilibrium equation system is obtained:

$$\begin{bmatrix} -c & 0 & 0 & -1 \\ s & 0 & 0 & 0 \\ c & -c & -c & 0 \\ -s & s & -s & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -2F \\ 0 \\ -F \end{bmatrix} = 0 \tag{2.21}$$

where $c = \cos \propto$ and $s = \sin \propto$.

In order to solve the matrix equilibrium equation system (2.4) we have to determine the inverse of the geometry matrix **G**, where $\alpha = 30°$ replaced into (2.21).

The steps of the solution are described in Table 2.1, where *Wolfram Mathematica* program applied.

```
s=Sin[Degree 30]
c=Cos[Degree 30]

q={0,-2F,0,-F}
G={{-c,0,0,-1},{s,0,0,0},{c,-c,-c,0},{-s,s,-s,0}}
MatrixForm[%]

IG=Inverse[G]
MatrixForm[%]
S=-IG.q
```

Table 2.1: Computation of internal forces of the 4-bar truss

The results as output of the solution are described in Table 2.2, obtained of computation process in Table 2.1 using *Wolfram Mathematica*.

The optimization problem given in formula (2.12 – 2.14) is already defined by the stress constraints. The maximal value of internal forces for cross section A_1 is $S_2 = 5F$ (see Table 2.2).

$\{4F, 5F, -F, -2\sqrt{3}F\}$

Table 2.2: Resulted internal forces of 4-bar truss

Therefore, the stress constraints will be the following:

$$\sigma_1 = 5F/A_1 \leq \sigma_0\,; \;\; \sigma_2 = 2\sqrt{3}F/A_2 \leq \sigma_0 \tag{2.22}$$

In order to solve the optimization problem dimensionless new variables are introduced.

$$X_1 = 5F/\sigma_0\, A_1\,; \;\; X_2 = 2\sqrt{3}F/\,\sigma_0\, A_2 \tag{2.23}$$

The optimization problem replaced by the new variables will be nonlinear in the objective function, but the obtained constraints become linear:

$$min\, f(X_1,\; X_2) = \rho\mathrm{LF}/\sigma_0\,(5/X_1 + 2/X_2\,) \tag{2.24}$$

$$X_1 \geq 0;\; ; X_2 \geq 0 \tag{2.25}$$

$$X_1 \leq 1; X_2 \leq 1 \tag{2.26}$$

The results of optimization problem in terms of X_1 and X_2, where only stress constraints are considered will be trivial solution, i.e. the maximal values are obtained for both variables (see in Table 2.3 and Table 2.4).

$\mathrm{Minimize}[\{5/\mathrm{X1} + 2/\mathrm{X2}\,, 0 \leq \mathrm{X1} \leq 1, 0 \leq \mathrm{X2} \leq 1\}, \{\mathrm{X1}, \mathrm{X2}\}]$

Table 2.3: Optimization of 4-bar truss replaced by new dimensionless variables

We are able to conclud as a statment, that the minimal weight design of a statically determinate structure subject to only stress constraints we obtain always trivial solution. But, subsequently we will see that when we extend the constraints by any others, the optimization results depend on the extended criteria namely we get an optima in intersection of the objective function and the function of the extended criteria.

$\{7, \{X1 \to 1, X2 \to 1\}\}$

Table 2.4: Results of optimization of 4-bar truss

2.3 Minimal Weight Design Subject to Stress and Displacement Constraints

In the following example we are looking for a minimal weight structure subject to stress and displacements constraints together. The problem solution is presented for the simple two-bar truss (see in Figure 2.3). However this structure statically determinate again the optimal solution will be already non-trivial. The geometry is given by the angle in between member 1 and member 2 ($\alpha = 30°$). The design variables are the area of the cross sections A_1 and A_2 that have to satisfy the stress limit σ_0 , the displacement limit u_0 , and the positivity criteria for the design variables.

The mathematical formulation of the optimization problem is described by the following way:

$$min\, f(A_1,\ A_2) = \rho L(A_1 + A_2/cos\alpha) \tag{2.27}$$

$$A_1 \geq 0;\ A_2 \geq 0 \tag{2.28}$$

$$\sigma_1 \leq \sigma_0\, ; \sigma_2 \leq \sigma_0 \tag{2.29}$$

$$u_1 \leq u_0\, ; u_2 \leq u_0 \tag{2.30}$$

Starting with the equilibrium equations of the internal forces:

$$\sum F_{ix} = 0, i = 1,2 \tag{2.31}$$

$$\sum F_{iy} = 0, i = 1,2 \tag{2.32}$$

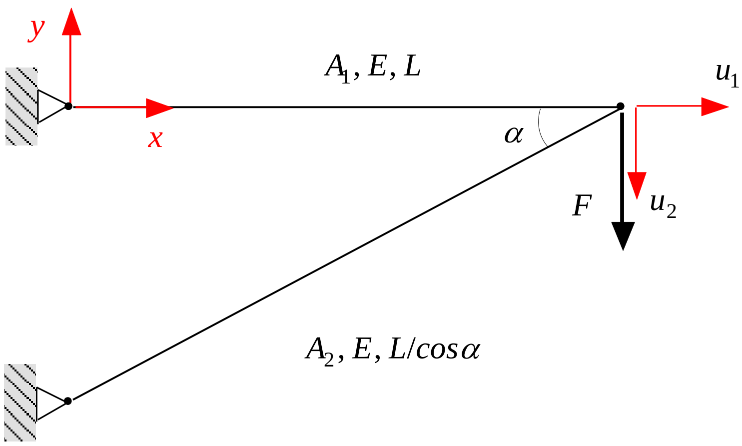


Figure 2.3 Initial data of two-bar truss optimization problem

Substituting the components of the internal forces into (2.30 and 2.31):

$$-S_1 - S_2 \cos \propto = 0 \tag{2.33}$$

$$-S_2 \sin \propto - F = 0 \tag{2.34}$$

In order to solve directly the equilibrium equations will be given in matrix form:

$$\begin{bmatrix} -1 & -c \\ 0 & -s \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -F \end{bmatrix} = 0 \tag{2.35}$$

where $c = \cos \propto$ and $s = \sin \propto$, and tension members are supposed.

The displacements are computed from the compatibility equation system, where linear elastic material applied i.e. the Hook's law is valid, the relationship in between the internal forces and the changes of the member length is given by the E elasticity modulus.

The compatibility equations are described in matrix form as well, where the displacements are transformed into local coordinate system:

$$\begin{bmatrix} -1 & 0 \\ -c & -s \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + {}^{L}\!/_{E} \begin{bmatrix} 1/A_1 & 0 \\ 0 & c/A_2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 0 \tag{2.36}$$

In order to determine the exact stress and displacement constraints we have to solve the (2.31 and 2.32) matrix equations. The steps of the solution are described in Table 2.5, where *Wolfram Mathematica* program applied:

```
Clear[c,s,q,G,IG,s,FLEX,TG,delta,ITG,u]
c=Cos[Pi/6]
s=Sin[Pi/6]
q={0,-F}

G={{-1,-c},{0,-s}}
MatrixForm[G]

IG=Inverse[G]
MatrixForm[IG]

s=-IG.q
FLEX=L/MOD*DiagonalMatrix[{1/A1,1/c/A2}]
TG=Transpose[G]
ITG=Inverse[TG]

delta=FLEX.s
u=-ITG.delta
```

Table 2.5: Computation of internal forces and displacements of the 2-bar truss[7]

$$\{\frac{\sqrt{3}FL}{\mathrm{A1MOD}}, -\frac{4FL}{\sqrt{3}\mathrm{A2MOD}}\}$$

$$\{\frac{\sqrt{3}FL}{\mathrm{A1MOD}}, -\frac{3FL}{\mathrm{A1MOD}} - \frac{8FL}{\sqrt{3}\mathrm{A2MOD}}\}$$

Table 2.6: Results of internal forces (see first row) and displacements (see second row) of 2-bar truss

[7] *Note: According to the special character regulation of the Wolfram Mathematica in the program the elasticity modulus (E) is replaced by the new name MOD.*

The resulted displacements vector will be given in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -FL/E \begin{bmatrix} -\sqrt{3}/A_1 \\ 3/A_1 + 8/\sqrt{3}A_2 \end{bmatrix}. \tag{2.37}$$

The displacement constraints is formulated for the vertical displacement of formula (2.37):

$$3FL/A_1 E + 8FL/\sqrt{3}A_2 E \leq u_0 \tag{2.38}$$

The stress constraints will be the following replased by the obtained internal forces:

$$\sigma_1 = \sqrt{3}F/A_1 \leq \sigma_0 ; \;\; \sigma_2 = 2F/A_2 \leq \sigma_0 \tag{2.39}$$

In order to simplify the optimization problem we will introduce again new variables instead of the original cross section variables A_1 and A_2.

The main goal of this reformulation of structural constraints is to create a linear explicit relationship in terms of new variables X_1 and X_2.

Let X_1 and X_2. In terms of variables A_1 and A_2.as following:

$$X_1 = \sqrt{3}F/A_1 \sigma_0 ; \;\; X_2 = 2F/A_2 \sigma_0 \tag{2.40}$$

Let the vertical displacement limit $u_0 = \sigma_0 L/E$, consequently, and the optimization problem is described in terms of X_1 and X_2 using the formula (2.40).

The optimization problem replaced by the new variables will be nonlinear in the objective function, but the obtained constraints become linear:

$$min\, f(X_1, \; X_2) = \mathrm{\rho LF}/\sigma_0 \, (3/X_1 + 4/X_2) \tag{2.41}$$

$$0 < X_1 \leq 1; \; 0 < X_2 \leq 1 \tag{2.42}$$

$$\sqrt{3}X_1 + \tfrac{4}{\sqrt{3}} X_2 \leq 1 \tag{2.43}$$

The results of optimization problem in terms of X_1 and X_2, where stress and displacements constraints are considered will be obtain using *Wolfram Mathematica* (see in Table 2.7) and the results of the solution is given in Table 2.8.

The grafical interpretation of the results is shown in Figure 2.3, where the optimal solution is obtined in the intersection of the linear displacement function and nonlinear objective function.

$$\text{NMinimize}[\{3/\text{X1} + 4/\text{X2}\,, \{0 \leq \text{X1} \leq 1, 0 \leq \text{X2} \leq 1, \text{Sqrt}[3] * \text{X1} + 4/\text{Sqrt}[3] * \text{X2} \leq 1\}\}, \{\text{X1}, \text{X2}\}]$$

Table 2.7: Optimization of 2-bar truss replaced by new dimensionless variables

$$\{28.290163223369824, \{\text{X1} \to 0.2474358293750296, \text{X2} \to 0.24743582935464875\}\}$$

Table 2.8: Results of optimization of 2-bar truss

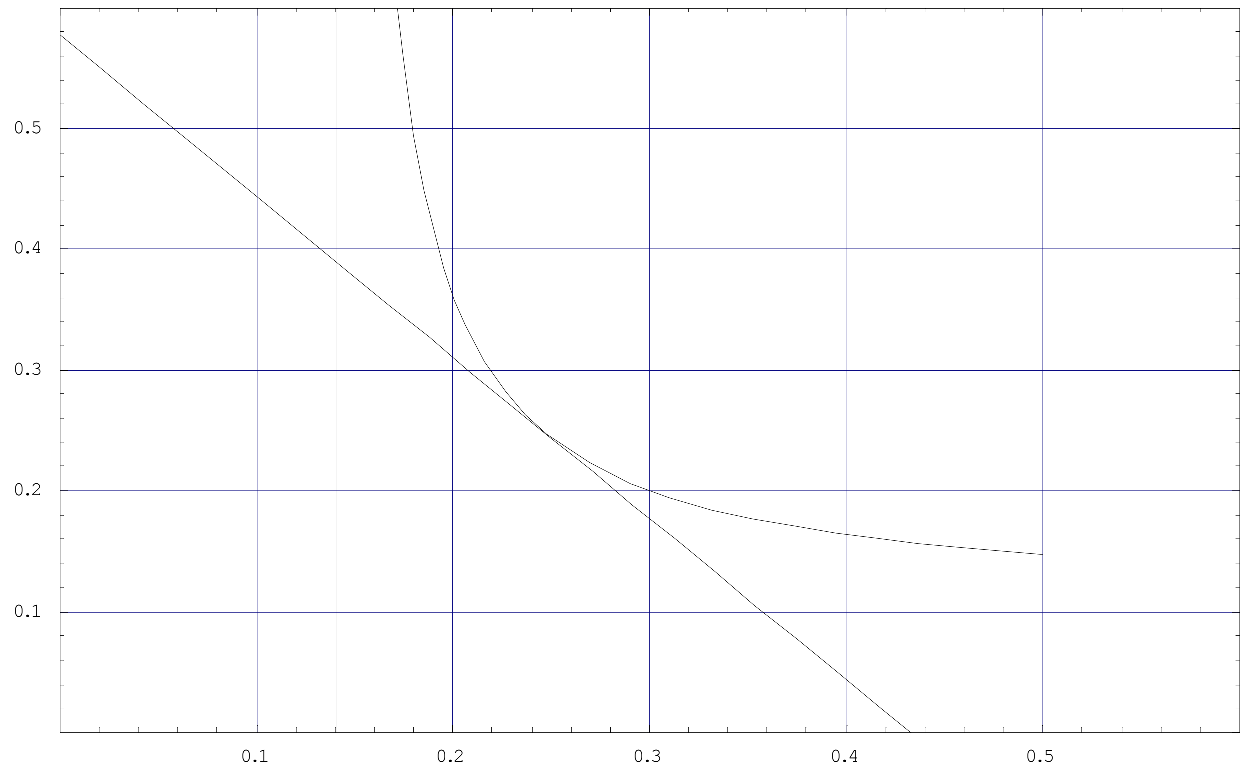


Figure 2.4 Results of two-bar truss optimization problem in terms of X_1 and X_2 design variables, where

$$\{\text{X1} \to 0.2474358293750296, \text{X2} \to 0.24743582935464875\}$$

The optimal values of cross sectional areas as the variables of the original optimization problem could be computed from formula 2.40, where the obtained X_1 and X_2, are replaced.

2.4 Minimal Weight Design Subject to Buckling Constraints

The two-bar truss (Fig. 2.5) minimal weight design is subjected to stress and member buckling constraints simultaneously. The given external vertical load causes tensile stress in member 1, and member 2 is under compression. The stress constraints for member 2 depend on the material law, the data of the geometry, the relationship in between of the cross section and the member length, and more over it depend on the intensity of applied external load. Therefore, we need to simplify some parameters to demonstrate a potential collapse mode.

The geometry is given by the angle in between member 1 and member 2 ($\alpha = 45°$). The design variables are the areas of the cross sections A_1 *and* A_2 that have to satisfy the stress limit σ_0, and the Euler buckling limit σ_b.

The mathematical formulation of the optimization problem is described by the following form:

$$min\, f(A_1,\ A_2) = \rho L(A_1 + A_2/cos\alpha) \tag{2.44}$$

$$A_1 \geq 0;\ A_2 \geq 0 \tag{2.45}$$

$$\sigma_1 \leq \sigma_0\, ; \sigma_2 \leq min\,[\sigma_0\, ;\ \sigma_b\,] \tag{2.46}$$

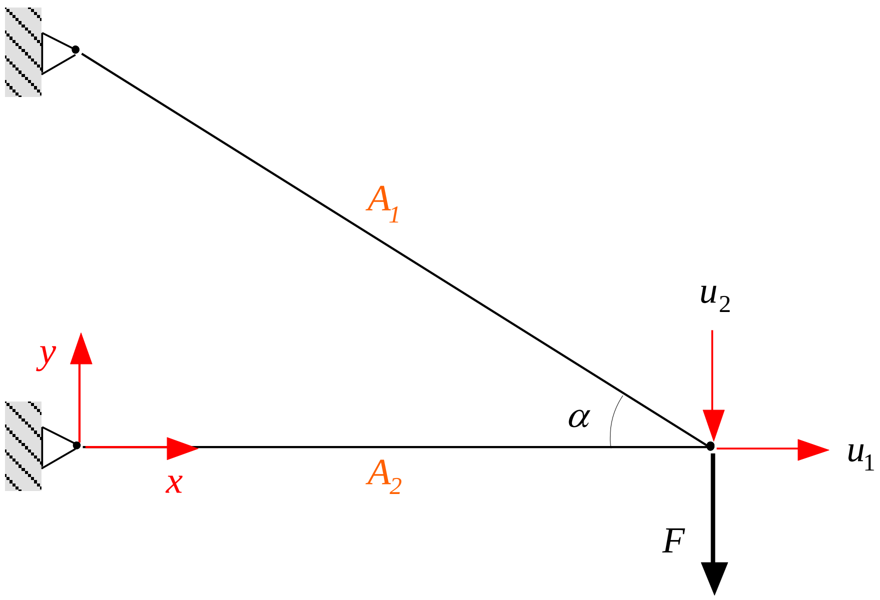


Figure 2.5 Initial data of two-bar truss optimization problem subject to buckling constraint

The equilibrium criteria are given in x, y coordinate system (see Fig. 2.5).

$$\sum F_{ix} = 0, i = 1{,}2 \tag{2.47}$$

$$\sum F_{iy} = 0, i = 1{,}2 \tag{2.48}$$

Substituting the components of the internal forces into formula (2.47) and (2.48):

$$-S_1 \cos \propto - S_2 = 0 \tag{2.49}$$

$$S_1 \sin \propto - F = 0 \tag{2.50}$$

where S_i , $i = 1{,}2$ are the internal forces of elements.

From the equilibrium equation system the following stress criteria are obtained:

$$A_1 \geq F \ \csc \alpha / \sigma_0 \tag{2.51}$$

$$A_2 \geq max \left[F \ \cot \alpha / \sigma_0 \, ; \ F \ \cot \alpha / \sigma_b\right] \tag{2.52}$$

The formula (2.52) contents already two different potential collapse cases depending on load intensity and the form of the cross section. In order to recognize the complexity of the buckling constraints the formula (2.52) is rewritten.

$$A_2 \geq max \left[F \ \cot \alpha / \sigma_0 \, ; \ F \ \cot \alpha / \sigma_b (A_2, I_2, L, E)\right] \tag{2.53}$$

where I_2 the moment of inertia of the cross section, E the elasticity modulus of the applied material.

In order to avoid further circumstances concerning instability against Euler buckling mode we suppose a circular cross section.

The buckling load for a hinged-hinged column is

$$P_b = EI\, \pi^2 / L^2 \tag{2.54}$$

where for a circular cross section

$$I = A_2^2 / 4\pi \tag{2.55}$$

The stress constraint against Euler critical load becomes

$$A_2^2 \geq 4FL^2 / \pi E \tag{2.56}$$

Example:

The relationship in between the load intensity factor and the radius of the applied cross section is demonstrated in Fig. 2.6.

In this example we supposed the following initial values: $\alpha = 45°$; $L = 100\ cm; E = 21000\ kN/cm^2$; $\sigma_0 = 24\ kN/cm^2$.

The intersection of these functions determines whether the buckling stress F_E or the stress limit F_H will be active during the structural optimization. According to the given actual initial values we could be determine the level of intersection ($F_E = F_H = 349.23\ kN$). Therefore, two load cases are distinguished:

- In case 1 ($F_1 = 400\ kN > 349.23\ kN$), the constraint of Euler-buckling doesn't active.
- In case 2 ($F_2 = 300\ kN < 349.23\ kN$), the constraint of Euler-buckling is active.

The results are obtained using Wolfram Mathematica numerical optimization procedure (In case 1 see in Table 2.9 and 2.10; and in case 2 see in Table 2.11 and 2 12).

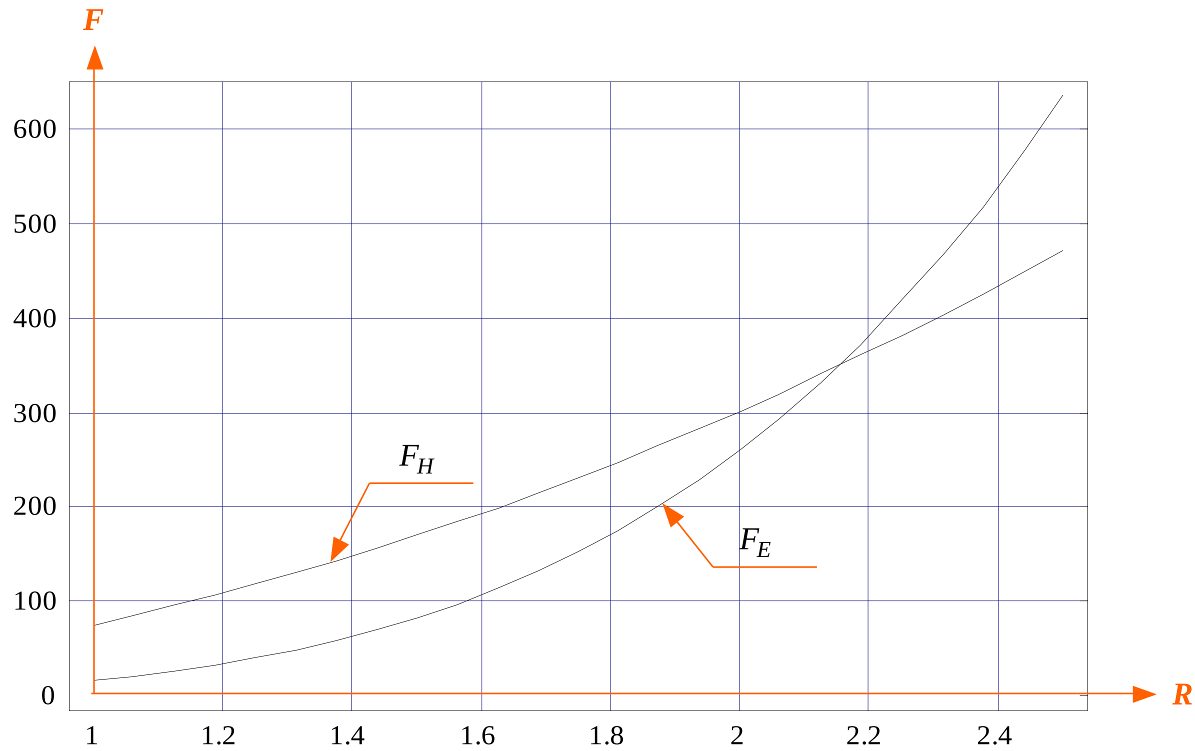


Figure 2.6 Changes of the stress constraints in the compression element in terms of the applied radius

The mathematical formulation of the optimization problem is described by the following form:

$$min\, f(A_1,\ A_2) = A_1\sqrt{2} + A_2 \tag{2.57}$$

$$A_1 \geq F\sqrt{2}/\sigma_0\,;\ A_2 \geq max[F/\sigma_0\,;\ 4FL^2/A_2\pi E] \tag{2.58}$$

$\text{NMinimize}[\{\text{A1}/c + \text{A2}, \text{A1} \geq 23.57023, \text{A2} \geq 16.6667\}, \{\text{A1}, \text{A2}\}]$
$\text{NMinimize}[\{\text{A1}/c + \text{A2}, \text{A1} \geq 23.57023, \text{A2} \geq 15.5731\}, \{\text{A1}, \text{A2}\}]$

Table 2.9: Optimization of 2-bar truss subject stress constraints

$\{50.0000389342532, \{\text{A1} \to 23.57023, \text{A2} \to 16.6667\}\}$
$\{48.9064389342532, \{\text{A1} \to 23.57023, \text{A2} \to 15.5731\}\}$

Table 2.10: Results of optimization of 2-bar truss subject sterss constraint

$\text{NMinimize}[\{\text{A1}/c + \text{A2}, \text{A1} \geq 17.6777, \text{A2} \geq 12.5\}, \{\text{A1}, \text{A2}\}]$
$\text{NMinimize}[\{\text{A1}/c + \text{A2}, \text{A1} \geq 17.6777, \text{A2} \geq 13.4867\}, \{\text{A1}, \text{A2}\}]$

Table 2.11: Optimization of 2-bar truss subject buckling constraints

$\{37.500043091562866, \{\text{A1} \to 17.6777, \text{A2} \to 12.5\}\}$
$\{38.486743091562865, \{\text{A1} \to 17.6777, \text{A2} \to 13.4867\}\}$

Table 2.12: Results of optimization of 2-bar truss subject buckling constraint

2.5 Minimal Weight Design of Three-Bar Truss Subject to Stress Constraints

The statically indeterminate three-bar truss presented in Fig. 2.7, have been considered by Uri Kirsch (see in (Kirsch, 1981); (Kirsch, 1993)) for minimal weight design subjected to stress and displacements constraints. In this chapter, in first step we will perform the computation of the structural constraints and the mathematical formulation will be defined for minimal weight design subject to only simple stress constraints.

Consider the three-bar truss shown in Fig. 2.7. The bars have Young's modulus E is given, let the angle for both inclined elements, α. The lengths of the elements are $L_1 = L/\cos\alpha\,; L_2 = L$; and $L_3 = L/\cos\alpha$, where a symmetrical geometry is supped.

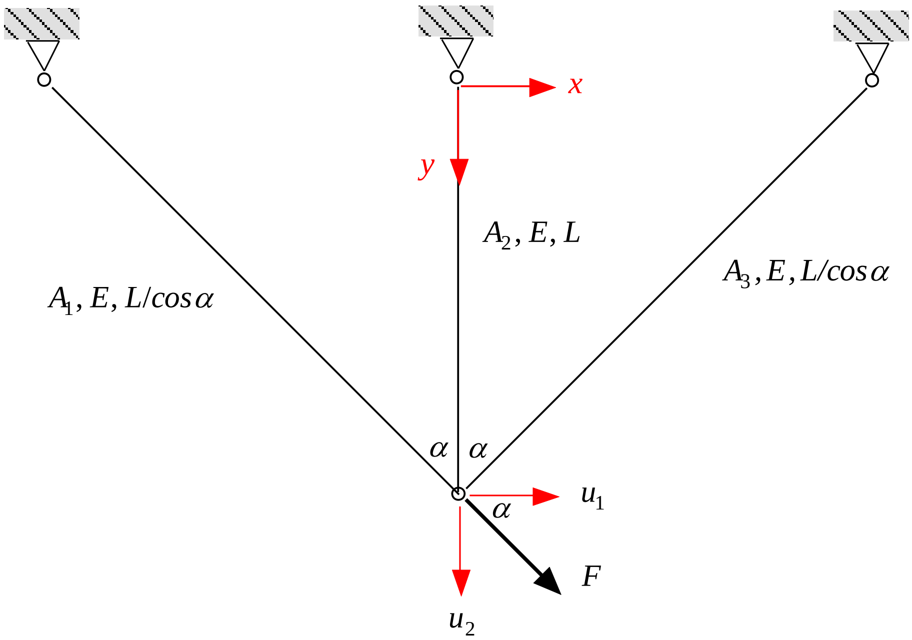


Figure 2.7 Initial data of the statically indeterminate three-bar truss.

The design variables are the cross-sectional areas $A1$, $A2$ and $A3$, but for simplicity we assume that the cross-sectional areas in bar-1 and bar-3 will be equal to each other ($A_3 = A_1$).

The objective function, which is the total weight of the truss, becomes

$$min\ f(A_1,\ A_2) = \mathrm{L}(\rho_1 A_1/cos\alpha + \rho_1 A_2 + \rho_3 A_1/cos\alpha) \tag{2.59}$$

where ρ_1, ρ_2 , and ρ_3 are the densities of the bars. The design constraints are

$$A_1 \geq 0;\ A_2 \geq 0 \tag{2.60}$$

$$|\sigma_i| \leq \sigma_0\,;\ i = 1,2,3 \tag{2.61}$$

The static equilibrium criteria are given in x, y coordinate system (see Fig. 2.7):

$$\sum F_{ix} = 0, i = 1,2,3 \tag{2.62}$$

$$\sum F_{iy} = 0, i = 1,2,3 \tag{2.63}$$

Substituting the components of the internal forces into formula (2.62) and (2.63):

$$-S_1 \cos \propto + S_3 \cos \propto + F \cos \propto = 0 \tag{2.49}$$

$$S_1 \sin \propto\ + S_2 + S_3 \sin \propto - F \sin \propto = 0 \tag{2.50}$$

where S_i , $i = 1,2,3$ are the internal forces of elements.

According to the statically indeterminate structure the formula (2.49) and (2.50) contains 3 unknown variables but we have only two equilibrium criteria available. In order to determine the stress constraint we need for other criteria. Let's consider the compatibility equilibrium criteria:

$$-u_1 \cos \propto + u_2\, sin \propto + LS_1/E\, A_1 \cos \propto = 0 \tag{2.51}$$

$$u_2 + LS_2/E\, A_2 = 0 \tag{2.52}$$

$$u_1 \cos \propto + u_2 \sin \propto + LS_3/E\, A_1 \cos \propto = 0 \tag{2.53}$$

The static and compatibility equilibrium together (2.49-2.53) already satisfy criteria of the solution for five unknowns, for $(S_1;\ S_2; S_3)$ internal forces and $(u_1;\ u_2)$ nodal displacements.

The matrix form of static and compatibility equilibrium (2.49-2.53) are obtained:

$$\begin{bmatrix} -\cos \propto & 0 & \cos \propto \\ \sin \alpha & 1 & \sin \alpha \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} + F \begin{bmatrix} \cos \propto \\ -\sin \propto \end{bmatrix} = 0 \tag{2.54}$$

$$\begin{bmatrix} -\cos \alpha & \sin \alpha \\ 0 & 1 \\ \cos \alpha & \sin \alpha \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + L/E \begin{bmatrix} 1/A_1 \cos \alpha & 0 & 0 \\ 0 & 1/A_2 & 0 \\ 0 & 0 & 1/A_1 \cos \alpha \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 0 \tag{2.55}$$

Starting from the static and compatibility equilibrium equation systems (2.54) and (2.55) the displacements based finite element method is obtained. The results of the nodal displacements is computed from the following matrix equilibrium equation

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{q} \tag{2.56}$$

where $\boldsymbol{K} = \boldsymbol{G}^T\boldsymbol{F}^{-1}\boldsymbol{G}$ is the stiffness matrix of the three-bar truss, $\boldsymbol{u}$ is the vector of the nodal displacements, and $\boldsymbol{q}$ is the load vector. The coefficient matrix of the displacements vector in formula (2.55) $\boldsymbol{G}^T$ is the coefficient matrix of internal forces in formula (2.54), and $\boldsymbol{F}^{-1}$ is the inverse matrix of the flexibility matrix in formula (2.55) supposed linear elastic material where the Hooke's law is valid.

The vector of the internal forces is obtained from $\boldsymbol{s} = -\boldsymbol{F}^{-1}\boldsymbol{G}\boldsymbol{u}$ is where $\boldsymbol{s}$ is the displacements vector, and $\boldsymbol{F}^{-1}\boldsymbol{G}$ is a part of the determination the stiffness matrix.

The main steps of the optimization design are presented in the subsequent pages where the angle is given ($\alpha = 45°$). Substituting into the matrix formulas (2.54) and (2.55):

$$\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} S_1 \\ S_1 \\ S_3 \end{bmatrix} + F \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 0 \tag{2.57}$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + L/E \begin{bmatrix} \sqrt{2}/A_1 & 0 & 0 \\ 0 & 1/A_2 & 0 \\ 0 & 0 & \sqrt{2}/A_1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 0 \tag{2.58}$$

The stiffness matrix is obtained from the multiplication of coefficient matrices $\boldsymbol{K} = \boldsymbol{G}^T\boldsymbol{F}^{-1}\boldsymbol{G}$

$$\boldsymbol{K} = E/L \begin{bmatrix} A_1/\sqrt{2} & 0 \\ 0 & A_1/\sqrt{2} + A_2 \end{bmatrix} \tag{2.59}$$

The resulted vector of displacements becomes

$$\boldsymbol{u} = FL/E \begin{bmatrix} 1/A_1 & 1/\left(A_1 + \sqrt{2}\, A_2\right) \end{bmatrix} \tag{2.60}$$

Finally, the vector of internal forces is computed from formula $\boldsymbol{s} = -\boldsymbol{F}^{-1}\boldsymbol{G}\boldsymbol{u}$ and resulted

$$\boldsymbol{s} = F\begin{bmatrix} (2A_1 + \sqrt{2}A_2)/2\left(A_1 + \sqrt{2}A_2\right) & A_2/\left(A_1 + \sqrt{2}A_2\right) & -A_2/\left(\sqrt{2}A_1 + 2A_2\right) \end{bmatrix} \tag{2.61}$$

The stresses of truss elements after some manipulation and simplification are the following

$$\sigma_1 = F\,(2A_1 + \sqrt{2}A_2)/\left(2A_1(A_1 + \sqrt{2}A_2)\right)$$

$$\sigma_2 = F/(A_1 + \sqrt{2}A_2)$$

$$\sigma_3 = -F\,A_2/A_1(\sqrt{2}A_1 + 2A_2)$$

In order to compare the stress constraints, we need to apply a common denominator, and obtained

$$\sigma_1 = F\,(A_2 + \sqrt{2}A_1)/(2A_1A_2 + \sqrt{2}A_1^2)$$

$$\sigma_2 = F\,\sqrt{2}A_1/(2A_1A_2 + \sqrt{2}A_1^2)$$

$$\sigma_3 = -F\,A_2/(2A_1A_2 + \sqrt{2}A_1^2)$$

In this example the densities of the bars will be equal to each other $\rho_1 = \rho_2 = \rho_3$. Let the stress limit for tension element $\sigma_0^+ = 20$, for compression element $\sigma_0^- = 20$, the member length $L = 100$, and the external load $F = 20$. Substituting the given values above into the formulas (2.59-2.61) the optimization problem is formulated following way:

$$min\, f(A_1,\ A_2) = 200\sqrt{2}A_1 + 100A_2 \tag{2.62}$$

$$A_1 \geq 0;\ A_2 \geq 0 \tag{2.63}$$

$$20\,(A_2 + \sqrt{2}A_1)/(2A_1A_2 + \sqrt{2}A_1^2) - 20 \leq 0 \tag{2.64}$$

$$20\,\sqrt{2}A_1/(2A_1A_2 + \sqrt{2}A_1^2) - 20 \leq 0 \tag{2.65}$$

$$-20\,A_2/(2A_1A_2 + \sqrt{2}A_1^2) - 15 \leq 0 \tag{2.66}$$

The results are obtained using Wolfram Mathematica numerical optimization procedure (see in Table 2.13 and 2.14).

A grafical interpretation of the optimal solution is presented in Fig. 2.8. In order to describe the intersection of the objective function and the function of the active constraint we have solve the equilibrium criteria for both functions (see in Table 2.15). The results is presented in Table 2.16.

```
Clear[A1, A2]

"Minimal Weight Design Subjected to Stress Constraints"

NMinimize[{200.∗ Sqrt[2] ∗ A1 + 100. A2, {Sqrt[2] + A2/A1 − 2.∗ A2 − Sqrt[2] ∗ A1 <= 0,

A1 >= 0, A2 >= 0}}, {A1, A2}]
```

Table 2.14: Optimization of optimization of 3-bar truss subject stress constraints

```
{263.89584336078366, {A1 → 0.7886751336089477, A2 → 0.40824829309546323}}
```

Table 2.14: Results of optimization of 3-bar truss subject stress constraints

```
NSolve[Sqrt[2] + A2/A1 − 2.∗ A2 − Sqrt[2] ∗ A1 == 0, {A2}]

NSolve[200. Sqrt[2] ∗ A1 + 100. A2 − 263.89584336078366 == 0, {A2}]
```

Table 2.15: Solution of objective function and the function of the active constraint

$$\{\{A2 \to \frac{-1.4142135623730951 + 1.4142135623730951A1}{-2. + \frac{1}{A1}}\}\}$$

$$\{\{A2 \to 0.01(263.89584336078366 - 282.842712474619A1)\}\}$$

Table 2.16: Results of objective function and the function of the active constraint in terms of A1

The graphical solution can be seen in Fig. 2.8 where the three nonlinear stress constraints and the linear objective function are presented. In Fig. 2.9, the optimal solution is given in the intersection of the optimal values of objective function and the only active stress constraint function (g_1).

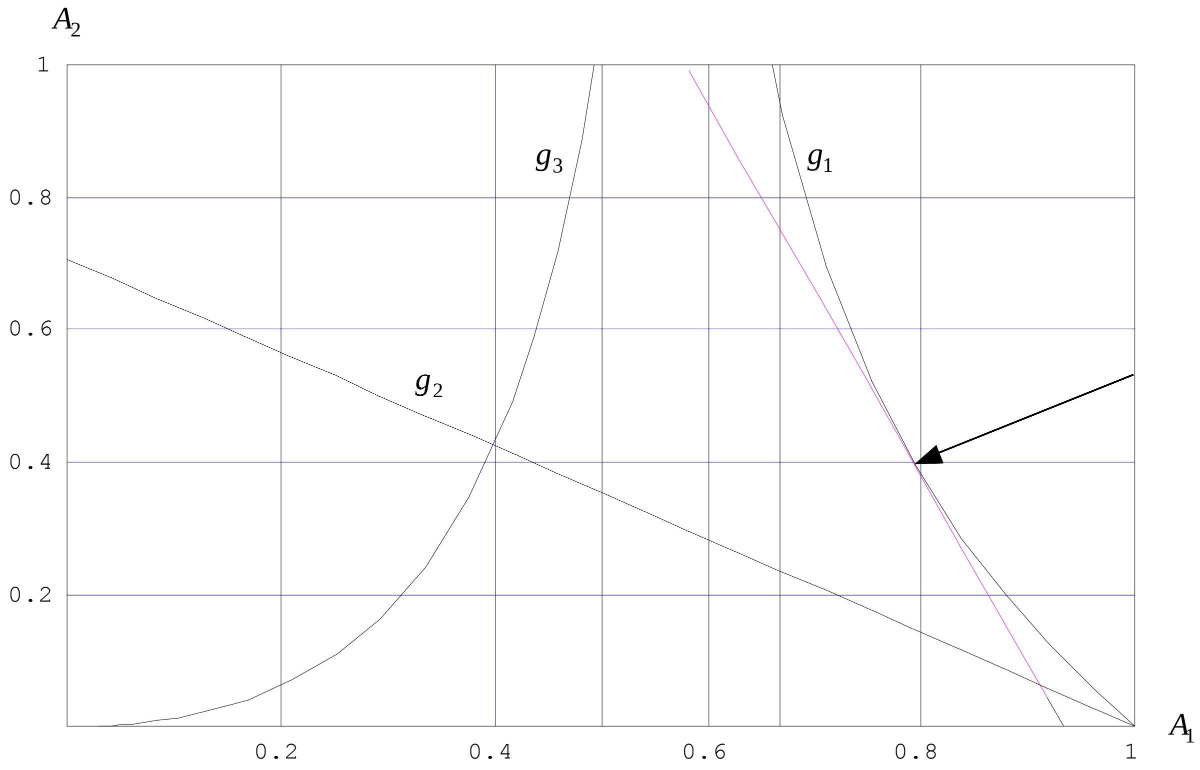


Figure 2.8 Grafical interpretation of the optimal solution with stress constraint functions

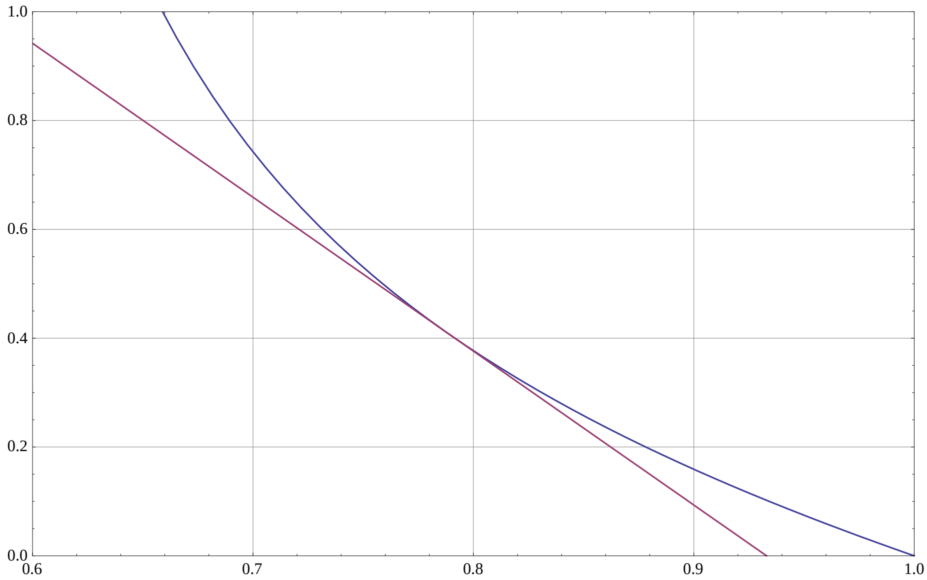


Figure 2.9 Grafical interpretation of the optimal solution. Intersection of the objective function and the function of the active constraint

2.6 Minimal Weight Design of Three-Bar Truss Subject to Stress and Displacement Constraints

In previous chapter the statically indeterminate three-bar truss has been considered for minimal weight design subjected to stress constraints. Before the mathematical formulation of the minimal weight optimization problem where only stress constraints were imposed the main steps of structural analysis was presented.

During the computation of the required internal forces the displacements are obtained as well solving the displacements based finite element equilibrium equation system. Considering the three-bar truss shown in Fig 10, we supposed that the bars properties are given. Let the Young's modulus (E), the angle for both inclined elements (α), and the lengths of the elements ($L_1 = L/\cos\alpha$; $L_2 = L$; $L_3 = L/\cos\alpha$) initially predicted, where a symmetrical geometry is supped.

In this chapter, we will present the effect of the displacements constraints to the final result obtained in chapter 2.5 (see Fig. 2.7). In order to compare the results computation of the minimal weight design subject to only simple stress constraints with the extended optimization problem where both the stress and displacements constraints are imposed we apply the same structural response data.

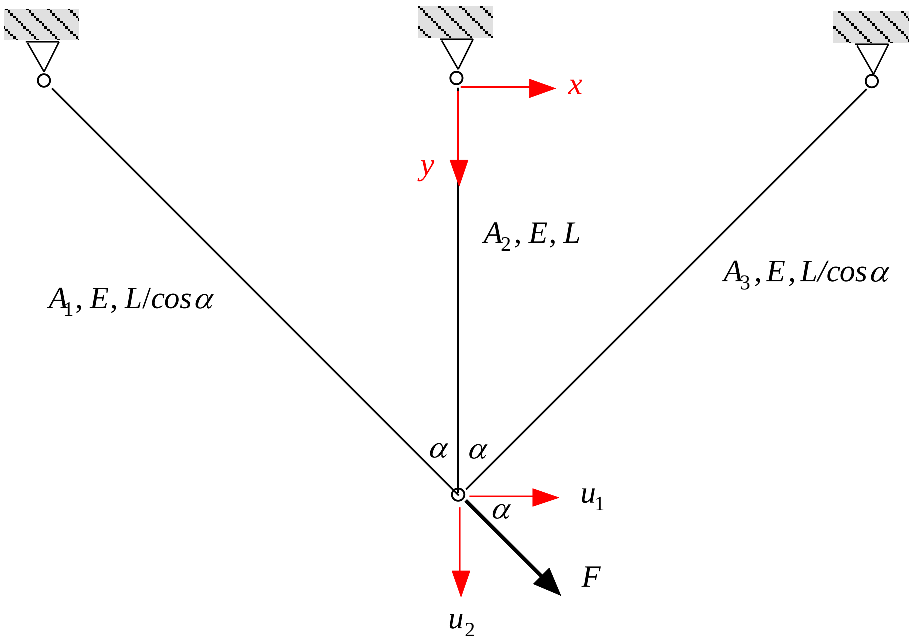


Figure 2.10 Initial data of three-bar truss optimization subjected to stress and displacement constraints.

In order to describe the mathematical formulation of the extended optimization problem the results of structural constraints (see 2.60) are presented here again, where the first element related to the horizontal, the second to the vertical displacement:

$$\boldsymbol{u} = FL/E\left[1/A_1 \quad 1/\left(A_1 + \sqrt{2}\,A_2\right)\right]$$

The stress constraints were already obtained in previous section as well:

$$\sigma_1 = F\,(A_2 + \sqrt{2}A_1)/\left(2A_1A_2 + \sqrt{2}A_1^2\right)$$

$$\sigma_2 = F\,\sqrt{2}A_1/\left(2A_1A_2 + \sqrt{2}A_1^2\right)$$

$$\sigma_3 = -F\,A_2/\left(2A_1A_2 + \sqrt{2}A_1^2\right)$$

In this example the mathematical formulation will be defined for minimal weight design subject to stress and vertical displacement constraints.

Let the densities of the bars will be equal to each other ($\rho_1 = \rho_2 = \rho_3$). Let given the stress limit for tension ($\sigma_0^+ = 20$), for compression ($\sigma_0^- = 15$), the member length $L = 100$, and the external load $F = 20$. In the former consideration the elasticity modulus hasn't role (see formulas 2.62-2.66) in optimization process, but with an extension by displacement constraint we need the exact value of elasticity modulus and displacement limit as well. Let elasticity modulus $E = 30000$ and the limit of vertical joint displacement $u_2^0 = 0.04$ is.

Substituting the given values above into the formulas the optimization problem is formulated:

$$min\, f(A_1,\ A_2) = 200\sqrt{2}A_1 + 100A_2$$

$$A_1 \geq 0;\ A_2 \geq 0$$

$$20\,(A_2 + \sqrt{2}A_1)/\left(2A_1A_2 + \sqrt{2}A_1^2\right) - 20 \leq 0$$

$$20\,\sqrt{2}A_1/\left(2A_1A_2 + \sqrt{2}A_1^2\right) - 20 \leq 0$$

$$-20\,A_2/\left(2A_1A_2 + \sqrt{2}A_1^2\right) - 15 \leq 0$$

$$2/30\left(A_1 + \sqrt{2}\,A_2\right) - 0.04 \leq 0$$

The results are obtained using Wolfram Mathematica numerical optimization procedure (see in Table 2.17 and 2.18).

A grafical interpretation of the optimal solution is presented in Fig. 2.8. In order to describe the intersection of the objective function and the function of the active constraints we have solve the equilibrium criteria for both functions (see in Table 2.19). The results is presented in Table 2.20.

```
Clear[A1, A2]
"Minimal Weight Design Subjected to Stress and Displacement Constraints"

NMinimize[{200.* Sqrt[2] * A1 + 100. A2, {Sqrt[2] + A2/A1 − 2.* A2 − Sqrt[2] * A1 <= 0,
2 − 30 * 0.04(A1 + Sqrt[2]A2) ≤ 0,
A1 >= 0, A2 >= 0}}, {A1, A2}]
```

Table 2.17: Optimizati of 3-bar truss subject stress and displacement constraints

```
{269.3740120760541, {A1 → 0.7142857111718958, A2 → 0.6734350404633282}}
```

Table 2.18: Results of optimization of 3-bar truss subject stress and displacement constraints

```
NSolve[Sqrt[2] + A2/A1 − 2.* A2 − Sqrt[2] * A1 == 0, {A2}]
NSolve[200. Sqrt[2] * A1 + 100. A2 − 269.374 == 0, {A2}]
NSolve[2 − 1.2(A1 + Sqrt[2]A2) == 0, {A2}]
```

Table 2.19: Solution of objective function and the function of the active constraint

$$A2 \to \frac{-1.4142135623730951 + 1.4142135623730951A1}{-2. + \frac{1}{A1}}$$

$$\{A2 \to 0.01(269.374 - 282.842712474619A1)\}$$

$$\{\{A2 \to -0.5892556509887896(-2. + 1.2A1)\}\}$$

Table 2.20: Results of objective function and the function of the active constraints in terms of A1

The graphical solution can be seen in Fig. 2.11 where the three nonlinear stress constraints, the linear displacement constraint, and the linear objective function are presented.

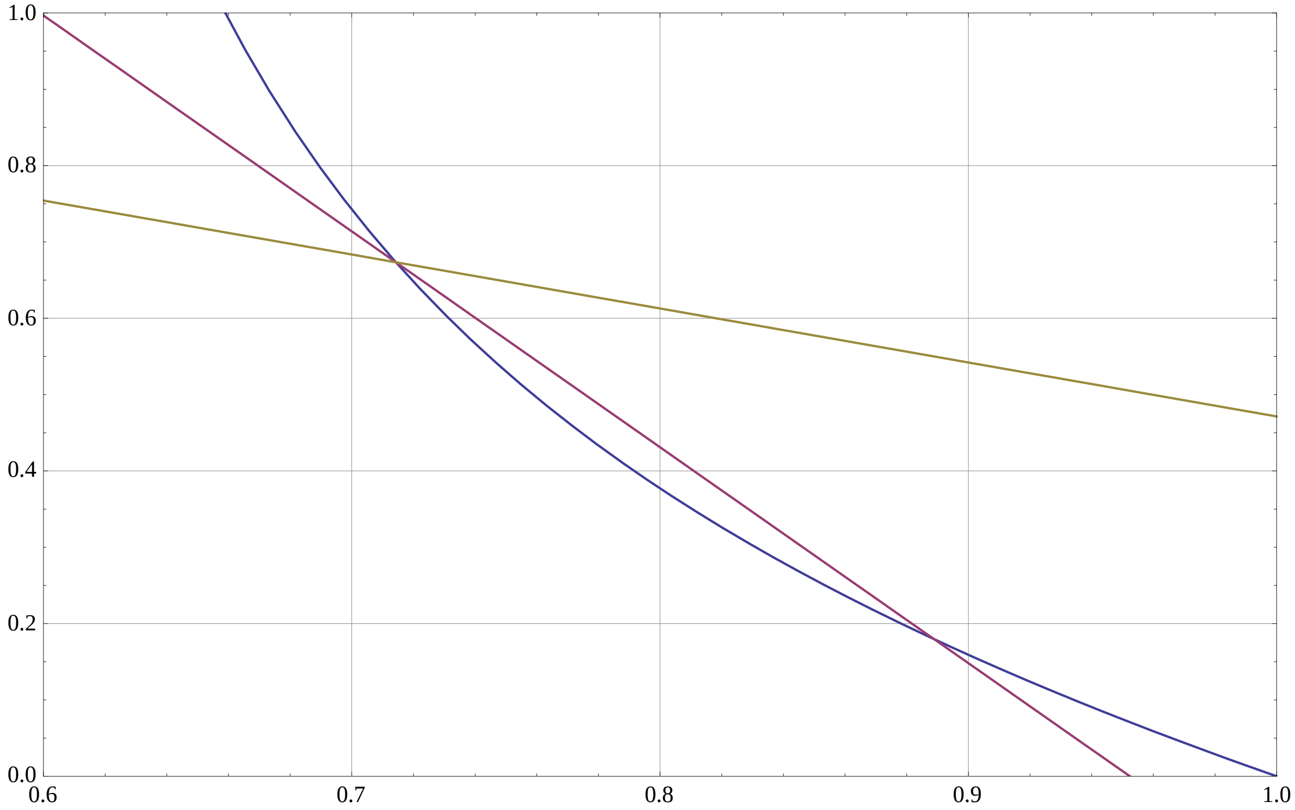


Figure 2.11 Grafical interpretation of the optimal solution with stress constraint functions

The optimal solution is given in the intersection of the optimal values of objective function and the active stress and displacement constraint function. In order to compare the difference in between two cases the results obtained subject to only stress constraints presented in Fig. 2.12.

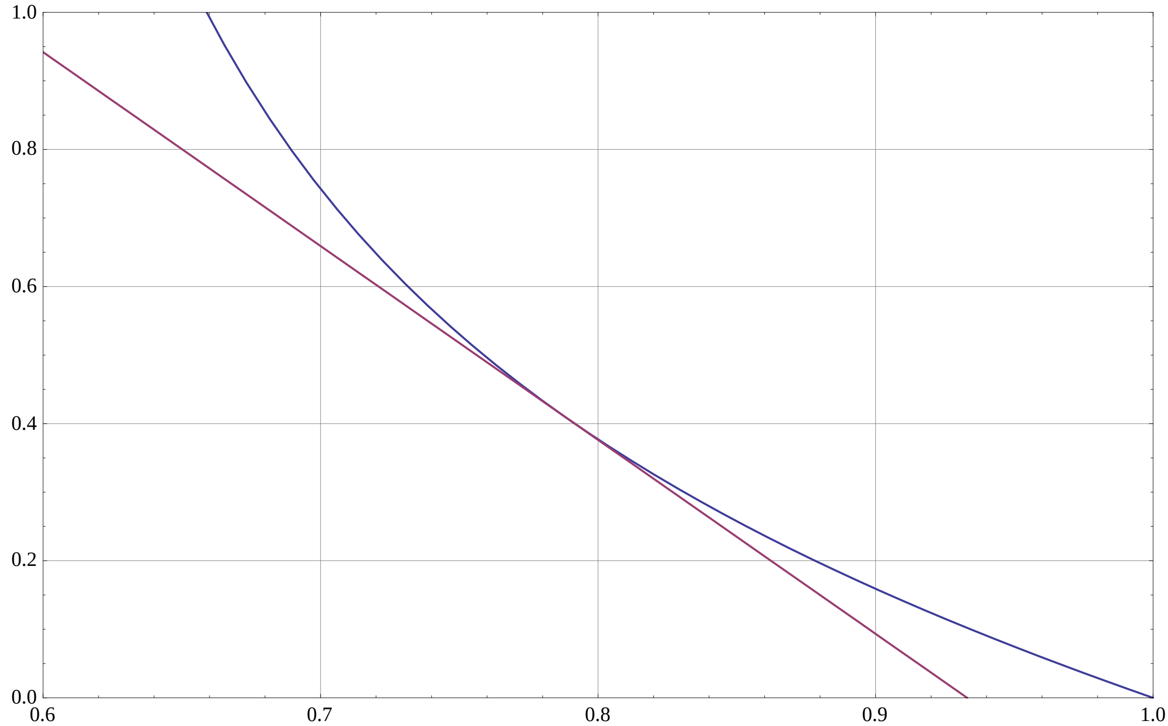


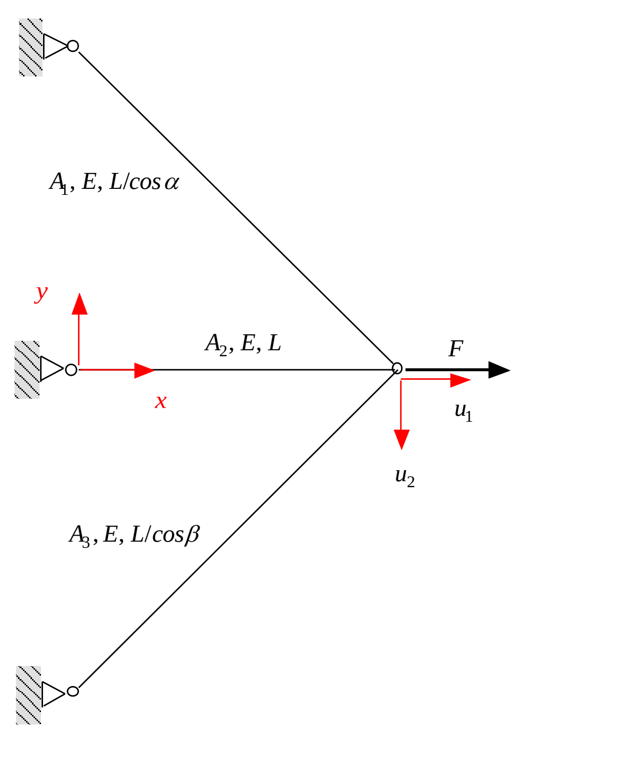
Figure 2.12 Grafical interpretation of the optimal solution. Intersection of the objective function and the function of the active constraint

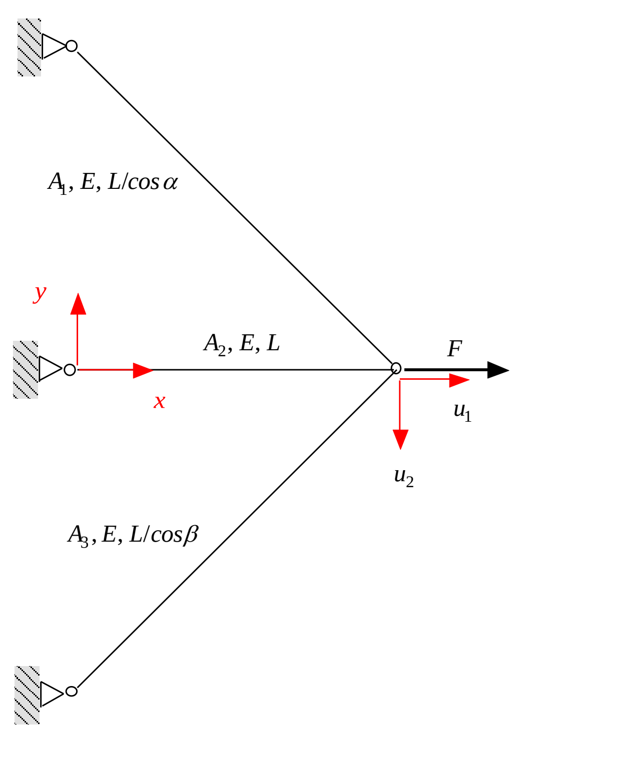
In case of extended version where the sterss and the vertical nodal displacement constraint are considered the obtained results will be $\mathrm{A1} = 0.7142857111718958$, $\mathrm{A2} = 0.6734350404633282$. The results can be seen in the intersection of the three functions, namely the first nonlinear stress constraint (blue line), the linear displacement constraint (green line), and the actual value of the objective function (red line).

2.7 Exercises

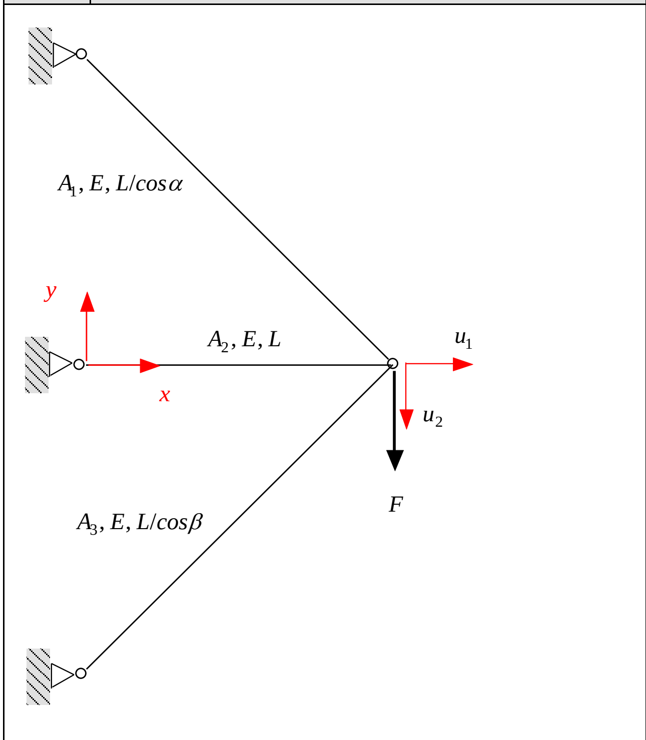
Determine the optimal solutions of the statically indeterminate three-bar truss in case A), B), C), and D) cases presented in Fig. 2.10. Describe the objective and constraint functions in 2D coordinate system and prove that only one will be active in optimal solution. Let the stress limit for tension element $\sigma_0^+ = 20$, for compression element $\sigma_0^- = 20$, the member length $L = 100$, and the external load length $F = 20$. Let the angle of elements $\alpha = \beta = 45°$ initially.

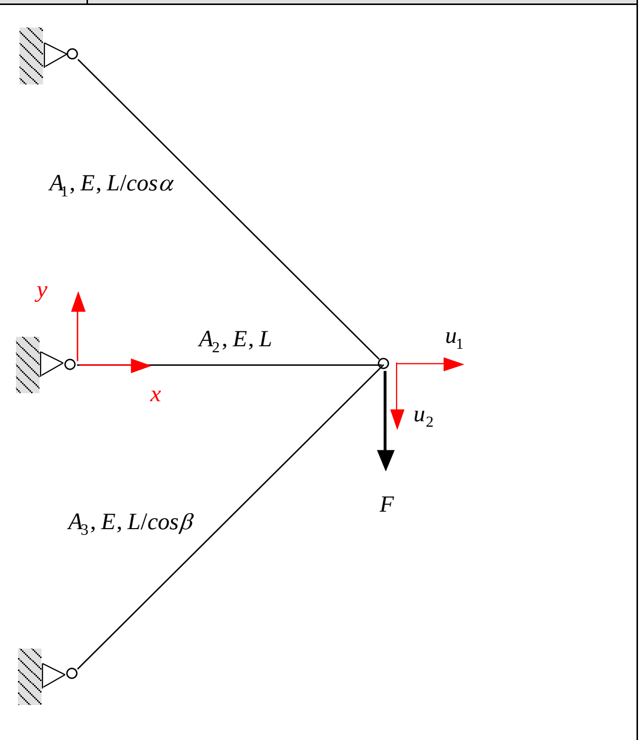
A	Let $\rho_1 = 2\rho_2 = \rho_3$; and $(A_3 = A_1)$.	B	Let $\rho_1 = 2\rho_2 = 2\rho_3$; and $(A_3 = A_1)$.





C	Let $\rho_1 = 2\rho_2 = \rho_3$; and $(A_3 = A_2)$.	D	Let $\rho_1 = 2\rho_2 = 2\rho_3$; and $(A_3 = A_2)$.





3. OPTIMIZATION OF ELASTO-PLASTIC STUCTURES

In the middle of last century has been appeared the theory of the plastic or elasto-plastic design. Drucker's (Drucker, 1958) paper of the "Variational principles in the mathematical theory of plasticity" and Koiter (Koiter, 1960) presentation about "General theorems for elastic-plastic solids" have been the basic theory of the application the elasto-plastic analysis in structural optimization.

Plastic design offers several advantages over the traditional elastic design. With plastic analysis, a structure can be designed to form a preselected yield mechanism at ultimate load level leading to a known and predetermined response during extreme events. The main advantage of plastic analysis that the optimal design problem can be cast in a linear programming (LP) form under the following assumptions (see (Kirsch, 1981)):

- Equilibrium conditions are referred to the undeformed geometry.
- The loads applied to the structure are assumed to increase proportionally.
- Constraints are related only to yield conditions and to design considerations.
- In trusses, it is required that the yield stress will not be exceeded in any member under any load condition.
- In frames, the magnitude of the bending moment in each cross section can at most be equal to the plastic moment. Linear relations between plastic moments as well as limitations on the plastic moments may be considered in the problem formulation
- The objective function represents the weight and can be expressed in a linear combination of the cross-sectional variables. Cross-sectional areas of truss members and plastic moments of frame members are chosen as design variables.

According to the strong simplifications assumed above it has been noted that "*The plastic analysis cannot replace the elastic analysis but supplements it by giving useful information about the collapse load and the mode of the collapse*"[8]. However, in special problems, such that e.g. large deformations, or instability consideration the plastic analysis seems very useful to give an estimated solution for further investigation of optimal structural design.

[8] Kirsch (1981) *Optimum Structural Design, Concepts, Methods and Applications*

For example, Maier (Maier, 1971) presented an incremental plastic analysis for large displacement structural model in order to tackle the instability phenomena. In this paper the plastic analysis of structures undergoing large deformations represents on the one hand a particularly difficult, still unsettled field of nonlinear mechanics, on the other hand an almost mandatory task for the engineer in many practical situations. It is not surprising therefore, that an intensive research effort has been and is being devoted to the many relevant problems. In the abundant literature available it seems possible to distinguish results mainly useful for the theoretical framework of the phenomenology in question and a stream of studies predominantly aimed at the numerical analysis of specific categories of structures[9].

3.1 Limit States

Those states are called limit states whenever they happen that the structure cannot satisfy the certain type of requirements prescribed by the standards. *The following limit states are classifying during the analysis and design of the elasto plastic structures* (see in (Lógó, 2012)):

Plastic limit state: the state when the entire elasto-plastic plastic structure or several members of it undergo unrestricted plastic deformation under constant external load is called the plastic limit state. The closed surface represented this state bounds the plastic limit load domain. It can be proved that this domain is convex seeing outwardly.

Shakedown limit state: Due to a variable, repeated multi-parameter loading there are generally unrestricted, accumulated plastic deformations. In connection with these plastic deformations, however, residual stresses also arise and can result in such a residual stress field which prevents the structure from undergoing any further plastic deformations during further repeated loading.

Plastic deformation state: in this state the plastic deformations are not accumulated and their magnitudes or any other measures reach a prescribed value.

Elastic limit state: one or several places of the structure plastic deformations occur at first simultaneously. The closed surface represented this state bounds the *elastic limit load domain*.

---

[9] Maier (1971) *Incremental plastic analysis in the presence of large displacements and physical instabilizing effects*

Elastic deformation state: all parts of the structure are in elastic state and the magnitudes of the elastic deformations and displacements reach an allowable value. The closed surface represented this state bounds the *elastic limited deformation domain*.

The goal of the analysis of the structures is to determine those conditions and bounds for the loading process at which the introduced limit states can be occurred. In case of one-parameter loading this goal is to determine the load multiplier.

- In the case of *analysis* it needs to investigate whether the loads exceed or not the loading domain prescribed in the standards or whenever they are happened they have enough safety reserve.
- In the case of *design* the materials and the dimensions of the structure should be determined on the way that the measure of the corresponding limit state of the structure subjected to the given load intensity does not exceed a limit given by the corresponding standards.

During the elasto-plastic analysis an idealized material property is assumed. The relationship in between of the stress and strain ($\sigma - \varepsilon$) is linear up to yield stress (see in Figure 3.1 a). Similar way the bending moment-rotation relation ($M - \theta$) will be perfectly linear elastic-plastic (Figure 3.1 b). At the fully plastic moment a plastic hinge is formed, and the curvature (rotation at the hinge) is increased without any increase in the moment The rotations at the cross section before M_P is reached are considered to be relatively small and the equilibrium equations are referred to the undeformed geometry of the structure. It is assumed that plastic hinges are concentrated at critical sections with ductility being unlimited. In addition, the loads applied to the structure are assumed to increase proportionally.

A *shakedown analysis* and *optimum shakedown design* of *elasto-plastic trusses* under *multi-parameter static loading* are presented in (Kaliszky, 2002). To control the plastic behaviour of the truss, bounds on the complementary strain energy of the residual forces and on the residual displacements are applied and for the bars under compression critical stresses updated during the iteration taken into consideration. The formulation of problems is suitable for nonlinear mathematical programming which is solved by the use of an iterative procedure. The application of the method is illustrated by three test examples.

In paper (Kaliszky, 2006) the authors pointed out that when in the design of structures *extreme loadings* such as short time, high intensity dynamic pressure (explosion), impact or earthquake have to be taken into consideration then, except for special cases, the plastic reserves of the material can be utilized, but the development of excessive plastic deformations, residual displacements and the collapse have to be prevented. Following this design concept in this paper three appropriate methods are presented for the determination of the optimal layout of material of elasto-plastic structures (beams, frames, trusses and plates) subjected to extreme loading. The investigation is extended also to the case when in the optimal design in addition to one of the extreme loads the normal (working) loads can be separately or simultaneously taken into consideration.

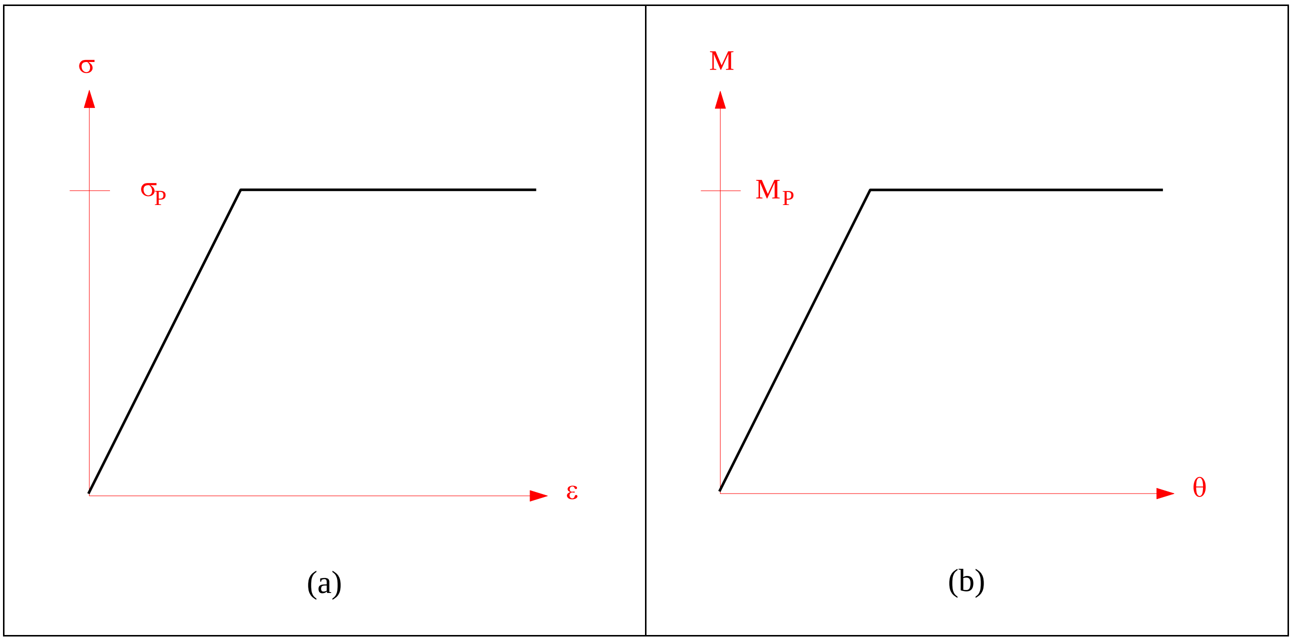


Figure 3.1 a) Idealized stress-strain relation, b) Idealized bending moment-curvature relation

Basically two types of existing methods for plastic analysis are distinguished on either the *kinematic approach* or the *static approach*[10]. There are many papers to demonstrate the advantages of the plastic design in case of different special problems (Kaliszky, 2006). The theoretical and analytical investigations more and more numerical methods and very accurate computer programs have been developed which can be used to a complete time history analysis of any kind of dynamically loaded elasto-plastic structures taking effects of viscosity and large deformation into consideration. The survey of these methods is beyond the scope of this paper. In the subsequent chapter, only *static approach* will be presented for two examples, which is often used in optimal design formulations.

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[10] Hodge P.G. Jr. (1959) *Plastic analysis of structures*, New York, McGraw-Hill Book Company

3.2 Plastic Analysis of Continuous Beam

Static approach could be applied for statically indeterminate structures. In this chapter the steps of the plastic analysis for the statically indeterminate continuous beam structure are considered shown in Fig. 3.2. The static equilibrium equations must be satisfied and the inequality conditions related to the given plastic capacities of critical sections. In this example a uniform plastic moment is supposed (M_P). According to the *static theorem of plastic analysis*[11], the bending moment distribution at collapse is such that the corresponding load factor is the largest *statically admissible* multiplier ($\lambda = \max(\lambda_i)$.). The problem of collapse load analysis will be considered under proportional loading.

Let the external loads given in one parametric form ($F_1 = \lambda F$; $F_2 = 2\lambda F$). Consequently, the bending-moment distribution is computed as shown in Figure 3.2.

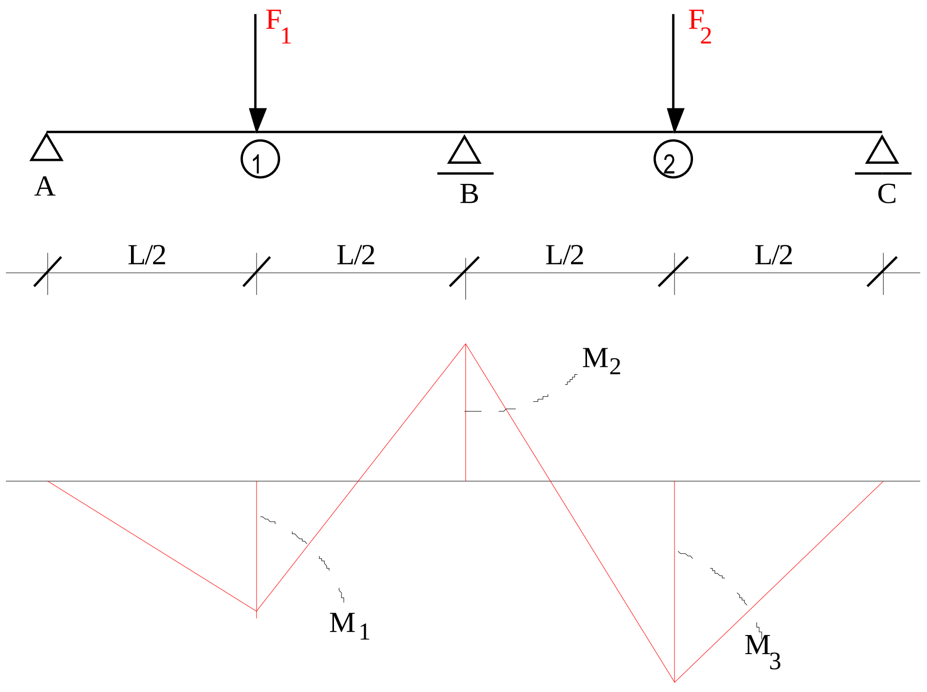

Figure 3.2 Plastic analysis of statically indeterminate beam structure

[11] Kirsch U. (1981) *Optimum Structural Design, Concepts, Methods and Applications*

The equilibrium equations are given terms of the support moment M_2 (see Fig. 3.2).

$$M_1 = \lambda \mathrm{FL}/4 - M_2/2 \tag{3.1}$$

$$M_2 = M_2 \tag{3.2}$$

$$M_3 = 2\lambda \mathrm{FL}/4 - M_2/2 \tag{3.3}$$

The number of critical sections is 3 but the number of equilibrium equations is only 2. The equations will be extended by the limit state requirements. Therefore, the 3 inequality yield conditions must be satisfied. The linear programming problem of plastic analysis is to find λ and M_j $(j = 1,2,3)$ such that

$$\lambda \rightarrow \max \tag{3.4}$$

$$4M_1 + 2M_2 = \lambda \mathrm{FL} \tag{3.5}$$

$$4M_3 + 2M_2 = 2\lambda \mathrm{FL} \tag{3.6}$$

$$-M_P \leq M_j \leq M_P \quad j = 1,2,3 \tag{3.7}$$

The solution of the linear programming problem is given in Wolfram Mathematica (see in Table 3.1). The results are presented in Table 3.2, and demonstrated the linear programming problem in Figure 3.3.

In order to simplify the problem description dimensionless new variables are introduced:

$$X_1 = \lambda \mathrm{FL}/M_P \tag{3.8}$$

$$X_2 = M_2/M_P \tag{3.9}$$

NMaximize[{X1, {−1 ≤ X2 ≤ 1, −1 ≤ (X1/4 − X2/2) ≤ 1, −1 ≤ (X1/2 − X2/2) ≤ 1}}, {X1, X2}]

Table 3.1: Plastic analysis of beam structure in terms of X_1 and X_2

{3., {X1 → 3., X2 → 1.}}

Table 3.2: Results of plastic analysis of beam structure in terms of X_1 and X_2

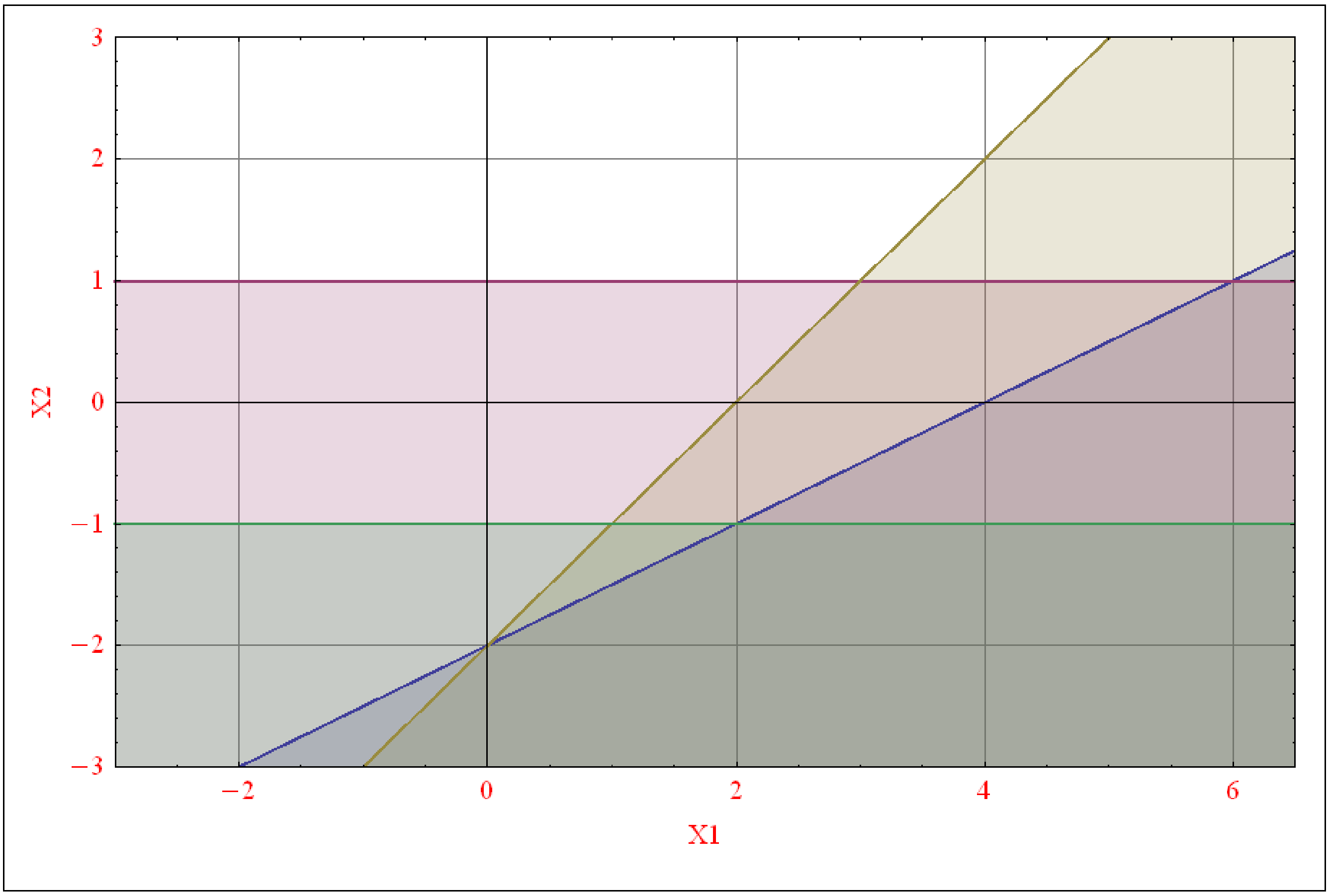

Figure 3.3: Graphical interpretation of the results of plastic analysis of beam in terms of X_1 and X_2[12]

The optimal solution is given in the intersection of the active constraint function $X_2 = 1$, and $X_2 = -2 + X_1$, in other words in the intersection of M_2 and M_3. Replaced by results of X_1 and X_2 in formula 3.8 and 3.9 the following values are obtained:

$$\lambda = 3M_P/\mathrm{FL} \tag{3.10}$$

$$M_1 = M_P/4 \tag{3.11}$$

$$M_2 = M_3 = M_P \tag{3.12}$$

[12] <u>Note</u>: *The colored area denotes the borders of feasible set.*

3.3 Plastic Analysis of Three-Bar Truss

Static approach could be applied for statically indeterminate trusses as well. In order to demonstrate the steps of the plastic analysis for the statically indeterminate three-bar truss structure are considered shown in Fig. 3.4. The static equilibrium equations must be satisfied and the inequality conditions related to the given plastic capacities of truss elements.

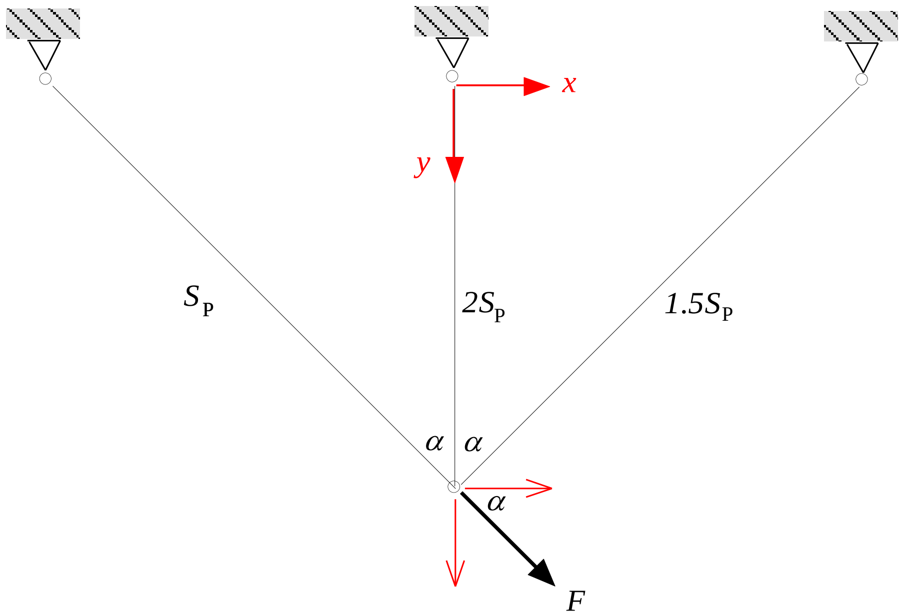

Figure 3.4 Plastic ultimate loads of three-bar truss elements.

The components of the internal forces are given in x, y coordinate system (see Fig. 3.4).

$$\sum F_{ix} = 0, i = 1,2,3 \tag{3.13}$$

$$\sum F_{iy} = 0, i = 1,2,3 \tag{3.14}$$

$$S_1/\sqrt{2} + S_2 + S_3/\sqrt{2} = \lambda F \tag{3.15}$$

$$S_1/\sqrt{2} - S_3/\sqrt{2} = 0 \tag{3.16}$$

The following plastic stress constraints must be satisfied the yield conditions simultaneously:

$$S_1 \leq S_P \tag{3.17}$$

$$S_2 \leq 2S_P \tag{3.18}$$

$$S_3 \leq 1.5S_P \tag{3.19}$$

The mathematical formulation of the plastic analysis is obtained by elimination of S_1 and S_3 using the equilibrium equations and substituting into the stress constraints.

Looking for the values of λ and S_2 such that:

$$\lambda \rightarrow \max \tag{3.20}$$

$$(\lambda F - S_2)/\sqrt{2} \leq S_P \tag{3.21}$$

$$S_2 \leq 2S_P \tag{3.22}$$

$$(\lambda F - S_2)/\sqrt{2} \leq 1.5S_P \tag{3.23}$$

The solution of the linear programming problem is given in Wolfram Mathematica (see in Table 3.3 and Table 3.4). In order to simplify the problem description dimensionless new variables are introduced:

$$X_1 = \lambda F/S_P \tag{3.24}$$

$$X_2 = S_2/S_P \tag{3.25}$$

NMaximize[{X1, {0 ≤ X2 ≤ 2,0 ≤ (X1 − X2)/Sqrt[2] ≤ 1,0 ≤ (X1 − X2)/Sqrt[2] ≤ 1.5}}, {X1, X2}]

Table 3.3: Plastic analysis of three-bar truss in terms of X_1 and X_2

3.414213562373095, {X1 → 3.414213562373095, X2 → 2. }

Table 3.4: Results of plastic analysis of three-bar truss in terms of X_1 and X_2

The results are presented in Figure 3.5, and demonstrated the linear programming problem.

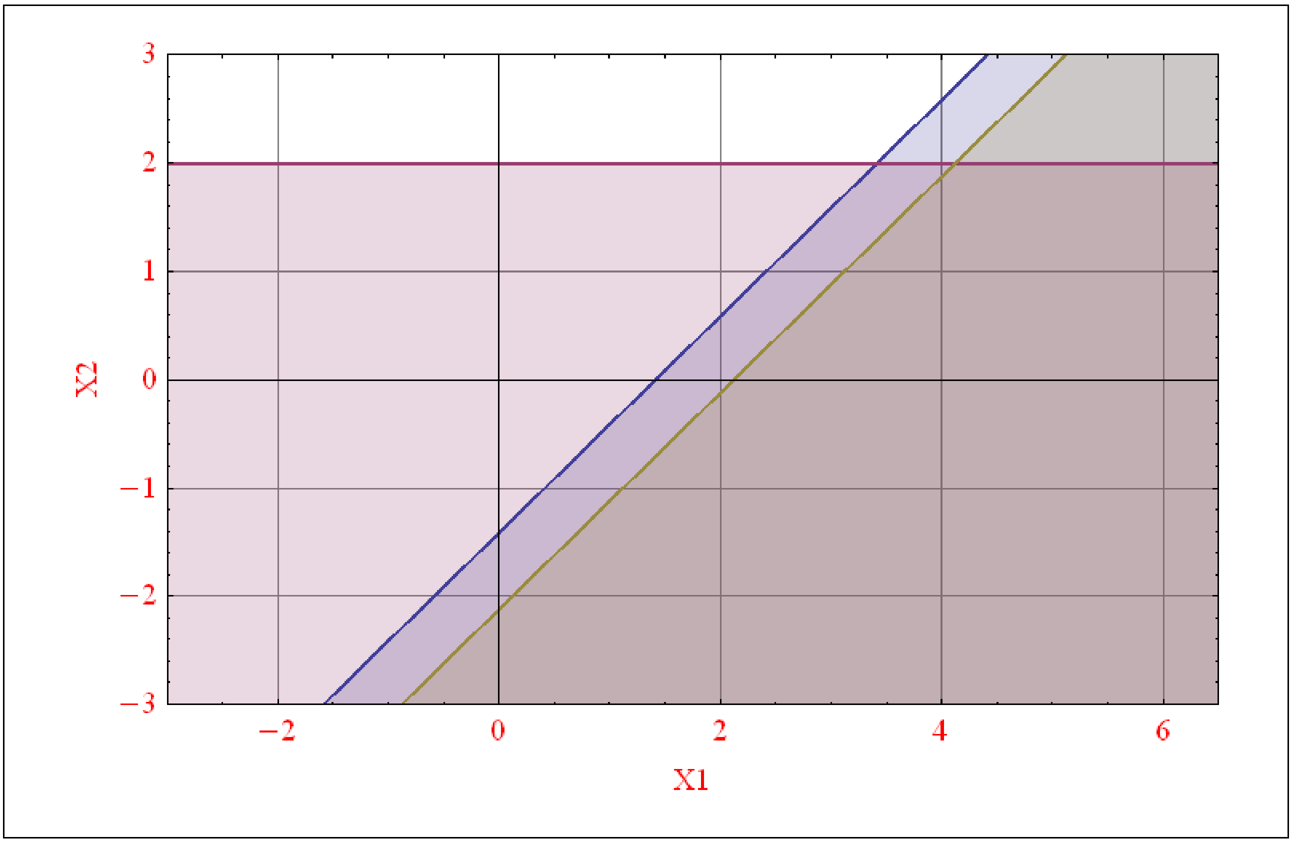

Figure 3.3: Graphical interpretation of the results of plastic analysis of three-bar truss in terms of X_1 and X_2

The optimal solution is given in the intersection of the active constraint functions $X_2 = 2$, and $X_2 = \left(-2 + \sqrt{2}\, X_1\right)/\sqrt{2}$, in other words in the intersection of S_1 and S_2. Replaced by results of X_1 and X_2 in formula 3.24 and 3.25 the following values are obtained:

$$\lambda = 3.4141 S_P/\mathrm{F} \tag{3.26}$$

$$S_2 = 2S_P \tag{3.27}$$

$$S_1 = S_3 = S_P \tag{3.28}$$

The results (3.26-3.28) and the Figure 3.3 demonstrate that the maximal load parameter is obtained when the internal loads in member 1 and 2 become equal to plastic limit simultaneously.

4. LINEAR PROGRAMMING FORMULATIONS

The available methods of structural optimization may conveniently be subdivided into two distinctly different categories called *analytical methods* and *numerical methods*. While analytical methods emphasize the conceptual aspect, numerical methods are concerned mainly with the algorithmical aspect.[13]

Analytical methods are usually employing the mathematical theory of calculus, in studies of optimal layouts or geometrical forms of simple structural elements. These methods are most suitable for such fundamental studies of single structural components, but are usually not intended to handle larger structural systems. The structural design is represented by a number of unknown functions and the goal is to find the form of these functions. The optimal design is theoretically found exactly through the solution of a system of equations expressing the conditions for optimality.

Numerical methods usually employ a branch in the field of numerical mathematics called *mathematical programming.* The recent developments in this branch are closely related to the rapid growth in computing capacities. In the numerical methods, a near optimal design is automatically generated in an iterative manner. An initial guess is used as a starting point for a systematic search for better designs. The search is terminated when certain criteria are satisfied indicating that the current design is sufficiently close to the optimum.

In the middle of the last century have been appeared the digital computer led to application of *linear programming* (LP) techniques to plastic design of frames (Hodge, 1959). This early numerical work is particularly significant in that it used mathematical programming techniques developed in the operations research community to solve structural design problems. During this period plastic design problems could be formulated as linear programming problems, and the application of mathematical programming techniques to structural optimization was limited to truss and frame structures. This type of structural optimization was focused primarily on steel frame structures and it did not consider stress, displacement or buckling constraints.

In the following pages some simple LP applications are presented using simplex method. The theory and its computation techniques are demonstrated and compared of computer results.

[13] Kirsch U. (1993) *Structural Optimization, Fundamentals and Applications,* Springer-Verlag

4.1 Simplex Method

In order to present the essence of the application of the simplex method first time an inequality problem is investigated.

We are seeking for the values of the defined variables such that satisfy the following inequalities:

$$3X_1 - X_2 + 2X_3 \le 7$$

$$2X_1 - 4X_2 \le 12$$

$$-4X_1 - 3X_2 + 8X_3 \le 10$$

Replace the former inequality constraints by an equality formulas extending with *slack variables* that will be used as starting point of the original inequality problem.

$$3X_1 - X_2 + 2X_3 + X_4 = 7$$

$$2X_1 - 4X_2 + X_5 = 12$$

$$-4X_1 - 3X_2 + 8X_3 + X_6 = 10$$

The equality equilibriums could be described in *basic formula* where we have to distinguish the *basic* and *non-basic* (*slack*) variables.

$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_0$
–	–	–	–	–	–	–
$3X_1$	$-X_2$	$+2X_3$	$+X_4$			$=7$
$2X_1$	$-4X_2$			$+X_5$		$=12$
$-4X_1$	$-3X_2$	$+8X_3$			$+X_6$	$=10$

Let a potential solution as an *initial point* of the original inequality problem:

$$X_1 = 0,\ X_2 = 0,\ X_3 = 0,\ X_4 = 7,\ X_5 = 12,\ X_6 = 10,$$

that will be described in the following vectorial formula:

$$7\mathbf{P}_4 + 12\mathbf{P}_5 + 10\mathbf{P}_6 = \mathbf{P}_0 \tag{4.1}$$

where $\mathbf{P}_4$, $\mathbf{P}_5$, $\mathbf{P}_6$ basis vectors are unit vectors.

Seeking for the next potential solution let introduce the basis vector $\mathbf{P}_1$, which resulted the following basis vectors:

$$3\mathbf{P}_4 + 2\mathbf{P}_5 + -4\mathbf{P}_6 = \mathbf{P}_1 \tag{4.2}$$

That is resulted:

$X_1^1 = 0,\ X_2^1 = 0,\ X_3^1 = 0,\ X_4^1 = 3,\ X_5^1 = 2,\ X_6^1 = -4$.

Multiply formula (4.2) by α and subtract from formula (4.1) we obtain:

$$(7 - 3\alpha)\mathbf{P}_4 + (12 - 2\alpha)\mathbf{P}_5 + (10 + 4\alpha)4\mathbf{P}_6 + \alpha\mathbf{P}_1 = \mathbf{P}_0 \tag{4.3}$$

Where value of α is computed as minimal quotient of initial and first step:

$\alpha = \alpha_0 = \min \dfrac{X_i^0}{X_i^1} = \dfrac{7}{3},\ i = 4,\ 5,\ 6$.

Replace α actual value into the formula (4.3) the first non-trivial solution is obtained:

$$\frac{22}{3}\mathbf{P}_5 + \frac{58}{3}\mathbf{P}_6 + \frac{7}{3}\mathbf{P}_1 = \mathbf{P}_0 \tag{4.4}$$

That resulted already a *non-zero basis vector*:

$X_1 = \dfrac{7}{3},\ X_2 = 0,\ X_3 = 0,\ X_4 = 0,\ X_5 = \dfrac{22}{3},\ X_6 = \dfrac{58}{3}$.

Continue the iteration solution process finally we obtain non-zero values for the originally defined real variables. Much efficient and expressive could be to give this elimination process in a table form.

Starting with the determination of minimal α value selected in last column ($\alpha = 7/3$) and the vector $\mathbf{P}_1$ will be selected as a new basis vector. The pivot row and pivot column is described in red frame.

$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_0$	α
3	-1	2	1	0	0	7	**7/3**
2	-4	0	0	1	0	12	12/2
-4	-3	8	0	0	1	10	

Table 4.1: Initial table

Square the pivot entry at the intersection of the pivot column and the pivot row, and identify entering variable and exit variable at mean time. Divide pivot by itself in that row to obtain 1. The further computation is presented in Table 4.2.

$\mathbf{P}_1^1$	$\mathbf{P}_2^1$	$\mathbf{P}_3^1$	$\mathbf{P}_4^1$	$\mathbf{P}_5^1$	$\mathbf{P}_6^1$	$\mathbf{P}_0^1$	α
1	-1/3	2/3	1/3	0	0	7/3	
0	-10/3	-4/3	-2/3	1	0	22/3	
0	-13/3	32/3	4/3	0	1	58/3	

Table 4.2: First non-trivial solution in terms of basis vectors $\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6$

The results are obtained in terms of basis vectors ($\mathbf{P}_1, \mathbf{P}_5, \mathbf{P}_6$) framed by blue color. The basis vectors $\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ are given in explicit formula as follows:

$$-\frac{1}{3}\mathbf{P}_1 - \frac{10}{3}\mathbf{P}_5 - \frac{13}{3}\mathbf{P}_6 = \mathbf{P}_2,$$

$$\frac{2}{3}\mathbf{P}_1 - \frac{4}{3}\mathbf{P}_5 + \frac{32}{3}\mathbf{P}_6 = \mathbf{P}_3,$$

$$\frac{1}{3}\mathbf{P}_1 - \frac{2}{3}\mathbf{P}_5 + \frac{4}{3}\mathbf{P}_6 = \mathbf{P}_4.$$

In the following subchapter a graphical interpretation will be considered in order to demonstrate the different options in selection of the “best” direction choice for basis change.

4.2 Linear Programming of 2D Problem

In this chapter already a two dimensional optimization problem is considered, where inequality constraints and a two dimensional objective function are given initially. The traditional simplex method described firstly. Subsequently, the problem solution is demonstrated in table form, and finally a graphical interpretation is presented to promote the understanding of the solution technique.

We are seeking for the maximal value of the two-dimensional objective function:

$$f(X_1, X_2) = 45X_1 + 80X_2$$

Such that satisfy the following inequality constraints:

$$5X_1 + 20X_2 \leq 400$$

$$10X_1 + 15X_2 \leq 450$$

$$X_1 \geq 0$$

$$X_2 \geq 0$$

The inequality constraints will be replaced by equality constraints such that the positivity criteria are kept in inequality condition.

$$5X_1 + 20X_2 + X_3 = 400$$

$$10X_1 + 15X_2 + X_4 = 450$$

$$X_i \geq 0,\ i = 1,\ 2.$$

The simplex method will be applied where the following basis vectors are considered:

$$\begin{array}{ccccc} \boldsymbol{P}_1 & \boldsymbol{P}_2 & \boldsymbol{P}_3 & \boldsymbol{P}_4 & \boldsymbol{P}_0 \\ - & - & - & - & - \\ 5X_1 & +20X_2 & +X_3 & & =400 \\ 10X_1 & +15X_2 & & +X_4 & =450 \end{array}$$

Let a potential solution as an *initial point* of the original LP problem

$X_1 = 0,\ X_2 = 0,\ X_3 = 400,\ X_4 = 450$,

that could be described by using basis vectors:

$$400\mathbf{P}_3 + 450\mathbf{P}_4 = \mathbf{P}_0 \tag{4.5}$$

where $\mathbf{P}_3$, $\mathbf{P}_4$ basis vectors are unit vectors.

Let introduced basis vector $\mathbf{P}_1$, according to the values of α ($\alpha = \frac{400}{5} = 80$, and $\alpha = \frac{450}{10} = 45$) the pivot element will be the second element of basis vector $\mathbf{P}_1$, that resulted:

$$5\mathbf{P}_3 + 10\mathbf{P}_4 = \mathbf{P}_1 \tag{4.6}$$

The following results will be obtained:

$X_1^1 = 0,\ X_2^1 = 0,\ X_3^1 = 5,\ X_4^1 = 10$.

Multiply formula (4.6) by α and subtract from formula (4.5) we obtain:

$$(400 - 5\alpha)\mathbf{P}_3 + (450 - 10\alpha)\mathbf{P}_4 + \alpha\mathbf{P}_1 = \mathbf{P}_0 \tag{4.7}$$

Replace α actual value into the formula (4.7) the first non-trivial solution is obtained:

$$175\mathbf{P}_3 + 45\mathbf{P}_1 = \mathbf{P}_0 \tag{4.8}$$

That led to the following results:

$X_1 = 45,\ X_2 = 0,\ X_3 = 175,\ X_4 = 0$.

The final results of the two-dimensional problem are obtained by the following series of the computational steps (see in Figure 4.1 and Table 4.3).

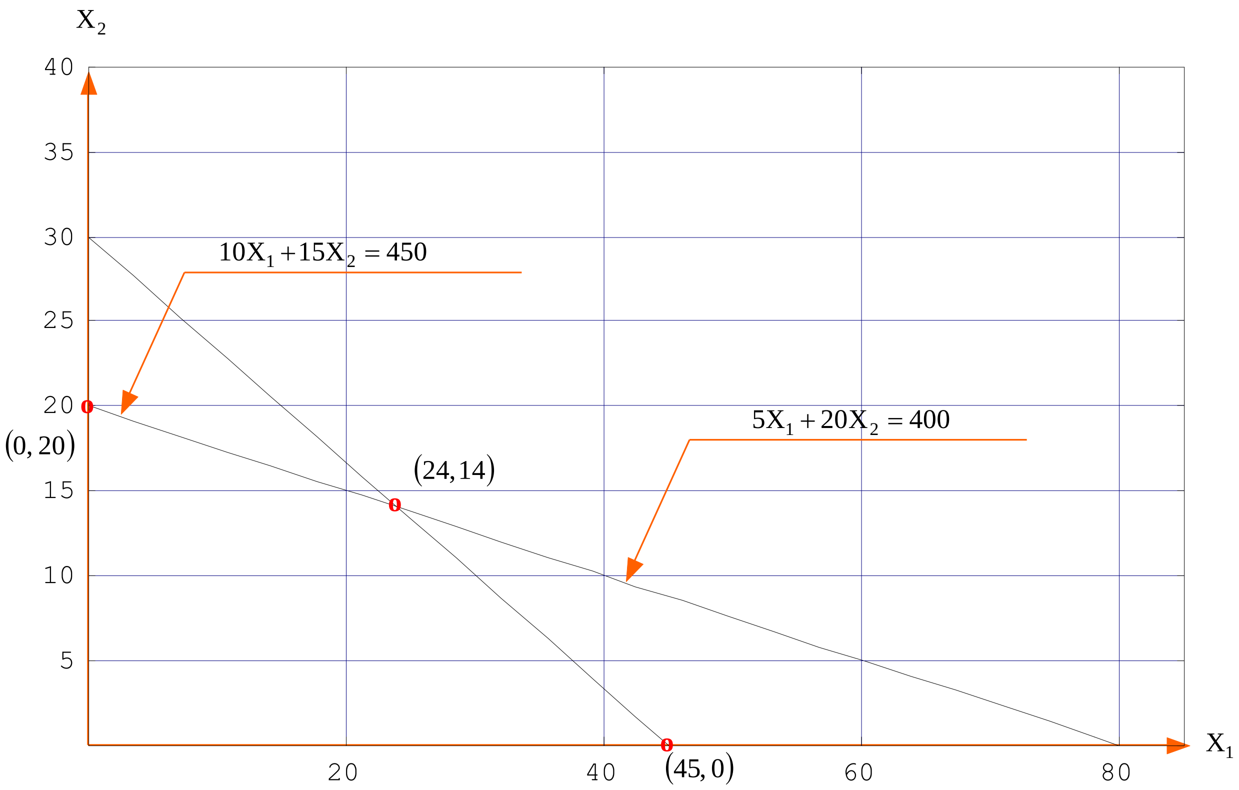

Figure 4.1: The graphical interpretation of the solution steps

X_1, X_2	Results	Basis equilibriums
$(0,0)$	$X_1 = 0,\ X_2 = 0,\ X_3 = 400,\ X_4 = 450$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 400 \\ 450 \end{bmatrix}$
$(45,0)$	$X_1 = 45,\ X_2 = 0,\ X_3 = 175,\ X_4 = 0$	$\begin{bmatrix} 5 & 1 \\ 10 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = \begin{bmatrix} 400 \\ 450 \end{bmatrix}$
$(0,20)$	$X_1 = 0,\ X_2 = 20,\ X_3 = 0,\ X_4 = 150$	$\begin{bmatrix} 20 & 0 \\ 15 & 1 \end{bmatrix}\begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = \begin{bmatrix} 400 \\ 450 \end{bmatrix}$
$(24,14)$	$X_1 = 24,\ X_2 = 14,\ X_3 = 0,\ X_4 = 0$	$\begin{bmatrix} 5 & 20 \\ 10 & 15 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 400 \\ 450 \end{bmatrix}$

Table 4.3: The steps of the solution of the two-dimensional problem

Subsequently, the problem solution is demonstrated in table form (Table 4.4-4.6).

$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_0^1$	α
5	20	1	0	400	400/5=80
10	15	0	1	450	450/10=45

Table 4.4: Initial table

The initial result is obtained in explicit formula as follows:

$X_1^1 = 0,\ X_2^1 = 0,\ X_3^1 = 5,\ X_4^1 = 10.$

$\mathbf{P}_1^1$	$\mathbf{P}_2^1$	$\mathbf{P}_3^1$	$\mathbf{P}_4^1$	$\mathbf{P}_0^1$	α
0	25/2	1	-1/2	175	350/25=14
1	3/2	0	1/10	45	90/3=30

Table 4.5: The first step resulted

The first non-trivial result is given in explicit formula:

$X_1 = 45,\ X_2 = 0,\ X_3 = 175,\ X_4 = 0.$

$\mathbf{P}_1^2$	$\mathbf{P}_2^2$	$\mathbf{P}_3^2$	$\mathbf{P}_4^2$	$\mathbf{P}_0^2$	α
0	1	2/25	-4/25	350/25	
1	0	-3/25	4/25	600/25	

Table 4.6: The second step resulted

The second step resulted already the final solution given in explicit formula:

$X_1 = 24,\ X_2 = 14,\ X_3 = 0,\ X_4 = 0.$

The solution method of the linear programming problem is given in Wolfram Mathematica (see in Table 4.7 and Table 4.8).

$$\mathrm{NMaximize}[\{45\mathrm{X1} + 80\mathrm{X2}, \{\mathrm{X1} > 0, \mathrm{X2} > 0{,}5\mathrm{X1} + 20\mathrm{X2} <= 400,$$
$$10\mathrm{X1} + 15\mathrm{X2} <= 450\}\}, \{\mathrm{X1}, \mathrm{X2}\}]$$

Table 4.7: The LP problem solution in terms of X_1 and X_2

$$\{2200., \{\mathrm{X1} \to 24.000000000000004, \mathrm{X2} \to 14.\}\}$$

Table 4.8: The results of LP problem in terms of X_1 and X_2

The graphical interpretation of the linear programming problem is computed in Wolfram Mathematica (see in Table 4.9 and Table 4.10) where the constraints and the objective function are solved in terms of X_2 .

$$\mathrm{Solve}[5 * \mathrm{X1} + 20 * \mathrm{X2} == 400, \mathrm{X2}]$$
$$\mathrm{Solve}[10 * \mathrm{X1} + 15 * \mathrm{X2} == 450, \mathrm{X2}]$$
$$\mathrm{Solve}[45\mathrm{X1} + 80\mathrm{X2} == 2200, \mathrm{X2}]$$

Table 4.9: The graphical interpretation of LP problem in terms of X_2

$$\mathrm{g1} = \frac{80 - \mathrm{X1}}{4}$$
$$\mathrm{g2} = -\frac{2}{3}(-45 + \mathrm{X1})$$
$$f = \frac{440 - 9 * \mathrm{X1}}{16}$$

Table 4.10: The functions of constraints and the objective of LP problem in terms of X_1

The functions of constraints and the objective function are described in Figure 4.2.

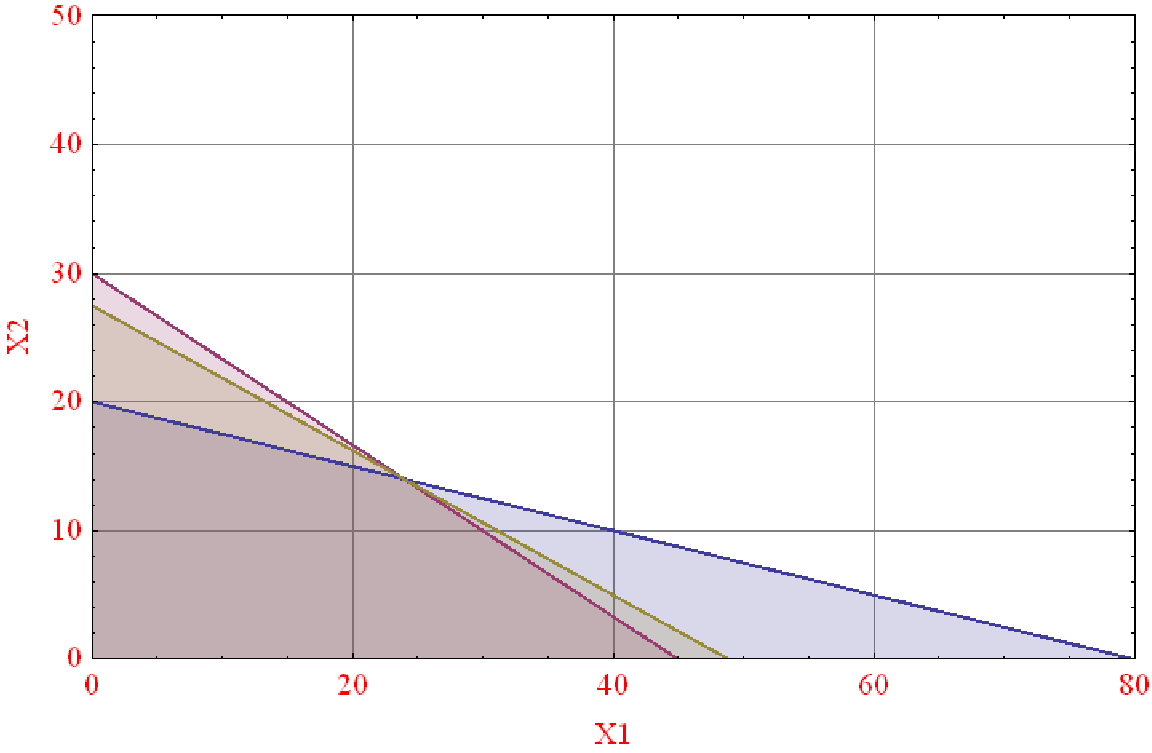

Figure 4.2: The graphical interpretation of LP problem

The final results of the two-dimensional problem are obtained using *Wolfram Mathematica* can be seen in intersection of the functions of constraints and the objective function where $X_1 = 24$, $X_2 = 14$, and the optimal value of the objective function $f(X_1, X_2) = 2200$.

4.3 Linear Programming of 3D Problem

Seeking for the maximal value of the following objective function:

$$f(x_1, x_2, x_3) = 3x_1 + x_2 + 3x_3$$

such that satisfy the next inequality constraints of the three-dimensional problems:

$$2x_1 + x_2 + x_3 \leq 2$$

$$x_1 + 2x_2 + 3x_3 \leq 5$$

$$2x_1 + 2x_2 + x_3 \leq 6$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

The table form of simplex method will be applied where the objective function is placed into the last row. In order to transform the originally maximization problem to minimization problem we have to apply opposite signs. The inequality constraints will be replaced by equality constraints. Let introduce the following three slack variables: x_4, x_5, x_6.

The values of the initial table are given as:

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, ekkor $x_4 = 2$, $x_5 = 5$, $x_6 = 6$.

Selecting the basic vector ($\mathbf{P}_2$), the pivot element will be the in the first row and second column see in Table 11.

$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_0$	α
2	1	1	1	0	0	2	2/1
1	2	3	0	1	0	5	5/2
2	2	1	0	0	1	6	6/2
-3	-1	-3	0	0	0	0	0

Table 4.11: Initial table

The next step is presented in Table 12.

$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_0$	α
2	1	1	1	0	0	2	2/1
-3	0	1	-2	1	0	1	1/1
-2	0	-1	-2	0	1	2	-
-1	0	-2	1	0	0	2	

Table 4.12: The second simplex table

In Table 12 the basic vector ($\mathbf{P}_3$) and the pivot element will be chosen in the second row.

The next step – where the pivot element will be chosen in first column and the first row – is presented in Table 14.

$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_0$	α
5	1	0	3	-1	0	1	1/5
-3	0	1	-2	1	0	1	-
-5	0	0	-4	1	1	3	-
-7	0	0	-3	2	0	4	

Table 4.13: The third simplex table

The final results are obtained in Table 14. According to the statement that in the last row no more negative sign values available, the elimination process doesn't resulted better results.

Therefore, the optimal solution is given as follows: $x_1 = 1/5$, $x_2 = 0$, $x_3 = 8/5$ $x_4 = 0$, $x_5 = 0$, $x_6 = 4$, where the value of the objective function is $f = 27/5$.

$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_0$	α
1	1/5	0	3/5	-1/5	0	1/5	
0	3/5	1	-1/5	2/5	0	8/5	
0	1	0	-1	0	1	4	
0	7/5	0	6/5	3/5	0	27/5	

Table 4.14: The fourth simplex table

Let prove the validity of the results of the LP problem presented above using in *Wolfram Mathematica* (see in Table 4.15 and Table 4.16). Comparing the results obtained using *Wolfram*

Mathematica we can be stated that we get exactly the same values for the objective function and the design variables as by stable simplex method.

$$\text{NMaximize}[\{3\text{X1} + \text{X2} + 3\text{X3}, \{\text{X1} >= 0, \text{X2} >= 0, \text{X3} >= 0, 2\text{X1} + \text{X2} + \text{X3} \leq 2,$$
$$\text{X1} + 2\text{X2} + 3\text{X3} \leq 5, 2\text{X1} + 2\text{X2} + \text{X3} \leq 6\}\},$$
$$\{\text{X1}, \text{X2}, \text{X3}\}]$$

Table 4.15: The LP problem solution in terms of X_1, X_2 and X_3

$$\{5.4, \{\text{X1} \rightarrow 0.2, \text{X2} \rightarrow 0., \text{X3} \rightarrow 1.6\}\}$$

Table 4.16: The results of the LP problem in terms of X_1, X_2 and X_3

4.4 Theory of Primal-Dual Linear Problems

In the previous chapter we presented that linear programming problems can be converted into an *augmented form* in order to apply the common form of the *simplex algorithm*. This form introduces non-negative *slack variables* to replace inequalities with equalities in the constraints.

Every linear programming problem, referred to as a *primal* problem, can be converted into a *dual problem*, which provides an upper bound to the optimal value of the primal problem.

According to the *theory of duality* every linear program is another called its dual. The dual of this dual linear program is the original linear program (which is then referred to as the primal linear program). Hence, linear programs come in primal/dual pairs. It turns out that every feasible solution for one of these two linear programs gives a bound on the optimal objective function value for the other ($min\ f(\boldsymbol{X}) = max\ g(\boldsymbol{W})$), where $\boldsymbol{X}$ is the vector of primal variables and $\boldsymbol{W}$ is the vector of the dual variables.

The original LP problem - called *primal problem* - can always be formulated as one of choosing vector of design variables ($X^T = [X_1 \quad X_2 \ldots \quad X_n]$) such that (Kirsch, 1993)

$$Z = \sum_{j=1}^{n} c_j X_j \rightarrow min \tag{4.9}$$

$$\sum_{j=1}^{n} a_{ij} X_j \geq b_i, i = 1, \dots, m \tag{4.10}$$

$$X_j \geq 0, j = 1, \dots, n \tag{4.11}$$

Note that we do not require $b_i \geq 0$, and that the equality constraints can also be expressed as inequalities.

$$\sum_{j=1}^{n} a_{kj} X_j = b_k \tag{4.12}$$

That can be replaced by the following two inequalities

$$\sum_{j=1}^{n} a_{kj} X_j \geq b_k, \quad -\sum_{j=1}^{n} a_{kj} X_j \geq -b_k \tag{4.13}$$

The *dual problem* of the previously defined *primal problem* (4.9-4.11) can be stated as follows

$$g(W) = \sum_{i=1}^{m} b_i W_i \rightarrow max \tag{4.14}$$

$$\sum_{j=1}^{m} a_{ij} W_i \geq c_j, j = 1, \dots, n \tag{4.15}$$

$$W_i \geq 0, i = 1, \dots, m \tag{4.16}$$

In matrix form, we can express the *primal – dual* formulation of linear programming problem as seen in Table 4.17:

THE PRIMAL PROBLEM	THE DUAL PROBLEM
$f(\boldsymbol{X}) = \boldsymbol{c}^T \boldsymbol{X}$ $\boldsymbol{X} = [X_1, X_2, \dots, X_n]$ $\boldsymbol{AX} = \boldsymbol{b}$ $\boldsymbol{X} \geq 0$	$g(\boldsymbol{W}) = \boldsymbol{W}^T \boldsymbol{b}$ $\boldsymbol{W} = [W_1, W_2, \dots, W_m]$ $\boldsymbol{A}^T \boldsymbol{W} = \boldsymbol{c}$

Table 4.17: The *primal – dual* formulation of the LP

4.5 The Dual Problem

In order to present the computation of a *primal – dual pair* let consider the example of chapter 4.2. Determine the dual problem and prove that the optimal solution of the objective function results exactly the same value. The formulation of the *primal – dual pair* is given in Table 4.18. The optimal solution is presented in Table 4.19 using *simplex method*.

THE PRIMAL PROBLEM	THE DUAL PROBLEM
$f(\boldsymbol{X}) = \boldsymbol{c}^T \boldsymbol{X} = -45X_1 - 80X_2 \Rightarrow min!$ $\boldsymbol{c}^T = [-45 \quad -80]$	$g(\boldsymbol{W}) = \boldsymbol{W}^T \boldsymbol{b} = 400W_1 + 450W_2 \Rightarrow max!$ $\boldsymbol{W}^T = ?$
$\boldsymbol{AX} = \boldsymbol{b},\ \boldsymbol{X} = [X_1, X_2, ..., X_n]$ $\begin{array}{llll} 5X_1 & +20X_2 & +X_3 & & = 400 \\ 10X_1 & +15X_2 & & +X_4 & = 450 \end{array}$ $\boldsymbol{X} \geq 0$	$\boldsymbol{A}^T\boldsymbol{W} = \boldsymbol{c},\ \boldsymbol{W} = [W_1, W_2, ..., W_m]$ $\begin{array}{lll} 5W_1 & +10W_2 & \leq -45 \\ 20W_1 & +15W_2 & \leq -80 \\ W_1 & & \leq 0 \\ & W_2 & \leq 0 \end{array}$

Table 4.18: The *primal – dual* formulation of the LP 4.2

The results of the primal problem is as follows $X_1 = 24,\ X_2 = 14,\ X_3 = 0,\ X_4 = 0$, the objective function $f(\boldsymbol{X}) = 2200$.

$\mathbf{P}_1^2$	$\mathbf{P}_2^2$	$\mathbf{P}_3^2$	$\mathbf{P}_4^2$	$\mathbf{P}_0^2$
0	1	2/25	-4/25	14
1	0	-3/25	4/25	24

Table 4.19: The simplex table of the LP 4.2

Solution of the *dual problem* using *simplex table* is presented in Table 4.20-Table 22.

W_1	W_2	W_3	W_4	c
5	10	1	0	45
20	15	0	1	80
-400	-450			

Table 4.20: The initial simplex table of the dual problem

W_1	W_2	W_3	W_4	c
0.5	1	0.1	0	4.5
12.5	0	-1.5	1	12.5
-175	0	45	0	2025

Table 4.21: The second simplex table of the dual problem

W_1	W_2	W_3	W_4	c
0	1	0.16	-0.04	4
1	0	-0.12	0.08	1
0	0	24	14	2200

Table 4.20: The final simplex table of the dual problem

The results of the dual problem is as follows $W_1 = 4$, $W_2 = 1$, $W_3 = 0$, $W_4 = 0$, the objective function $f(\boldsymbol{X}) = 2200$. We have to note that the last row contents the solution of the primal problem, in the column of slack variables: $X_1 = 24$, $X_2 = 14$.

It is *already stated* that the primal - dual pair of an LP results the *same value* for the *objective function*. Moreover *it is proven* that enough to solve one problem using simplex table because the *results for both* problems *can be seen in the final simplex* table.

5. OPTIMIZATION SUBJECT TO STRUCTURAL INSTABILITY

The constrained truss optimization where *structural instability* is considered simultaneously requires the solution of the *equilibrium path* and the detection of *critical points* on the equilibrium path i.e. *bifurcation* or *limit points*. The simple iteration strategies for computation of the equilibrium path do not provide an accurate tool for the stability point computation since the basis is an incremental procedure. Therefore, in the neighborhoods of the singular points the simple iteration process became unstable.[14]

To avoid these difficulties, the *arc length procedures* or *continuation methods* are applied in the finite element literature, e.g. (Criesfield, 1981), (Schweizerhoff, 1986). The basis for this procedure is an extension of the nonlinear set of equilibrium equations by a constraint condition that introduces additional information about stability points. Such extended systems facilitate the computation of *limit or bifurcation points* directly, e.g. (Wriggers, 1988). In this chapter a higher order path-following method (Csébfalvi, 1998) is proposed based on the perturbation technique of the stability theory and a non-linear modification of the classical linear homotopy method.

The *nonlinear function* of the *total potential energy* for conservative systems can be expressed in terms of *nodal displacements* and the *load parameter*. The *equilibrium equations* are given from the *principle of stationary value of total potential energy*. The *stability investigation* is based on the *eigenvalue analysis of the Hessian matrix*. In each step of the *path-following* process, we get information about the *displacement, stresses, local, and global stability* of the structure. With the help of the *higher-order predictor-corrector* algorithm, we are able to compute an arbitrary load deflection path and detect the different type of stability points. During the optimization process, a truss design is characterized by its *maximal load intensity factor* along the equilibrium path.

The proposed *third order predictor-corrector* method has been successfully used for several problems. The shallow truss dome problem is a very good example of stability loss. The dome of Figure 5.1, if subjected to vertical forces at the given nodes, deforms until it loses stability at the first *turning point*. After passing this point, the dome exhibits a more complicated behavior, when we consider the global snap-through.

[14] Csébfalvi A. (1998) *A nonlinear path-following method for computing the equilibrium curve of structures*, Annals of Operations Research, 81, 15-24

5.1 Stationary Principle of Potential Energy

The optimization subjected to *stability constrains* requires a *nonlinear consideration* of the structural behavior based on the *principle of minimum potential energy function*.

In this chapter a nonlinear path-following method is given for the analysis of nonlinear instability problems. The proposed method is based on the perturbation technique of stability theory (see (Koiter, 1960) and (Thompson, 1984)), and on the nonlinear modification of the classical linear homotopy method. The essence of the classical homotopy method is to reduce the solution of nonlinear equilibrium equation to that of first-order implicit ordinary differential equations. The introduction of *higher-order prediction* makes it possible to approximate the curve of the *equilibrium path*, so that the proposed *pathfollowing method* is able to compute the information of direct methods and then compute not only points but also segments of the equilibrium path. The curve approximation simplifies the computation of stability points.[15]

The nonlinear function of the total potential energy for conservative systems can be expressed in terms of nodal displacements and the load parameter:

$$V(u_i, \lambda), i = 1, 2, \dots, N \tag{5.1}$$

where $V(u_i, \lambda)$ is the nonlinear total potential energy function, $u_i, , i = 1, 2, \dots, N$ the vector of the nodal displacements, the λ is the load intensity factor, and N is the degree of the freedom.

The equilibrium equations are given from the principle of stationary value of total potential energy:

$$V_{,i}(u_i, \lambda) = 0, \ i = 1, 2, \dots, N \tag{5.2}$$

where the subindices $()_{,i}$ of V following the coma denote the partial derivatives by the displacement $u_i, , i = 1, 2, \dots, N$.

With the introduction of a common notation of nodal displacements and the load intensity factor, the equilibrium equations are replaced by the following formula.

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[15] Csébfalvi A. (1998) *A nonlinear path-following method for computing the equilibrium curve of structures*, Annals of Operations Research

$$V_{,i}(y_k) = 0, \; k = 1, 2, \dots, N + 1 \qquad (5.3)$$

where (y_k)'s signify the N +1 dimensional solution vector.

The stability investigation is based on the eigenvalue computation of the Hessian matrix. In each step of the *path-following process*, we get information about the displacement, stresses, local, and global stability of the structure.

The theoretical background of the instability computation is presented in (Csébfalvi, 1998). This paper prove that the introduction of higher-order prediction makes it possible to approximate the curve of the equilibrium path, so that the proposed pathfollowing method is able to compute the information of direct methods and then compute not only points but also segments of the equilibrium path. The curve approximation simplifies the computation of stability points.

Within the predictor step, we compute the solution of an implicit ODE problem. The corrector phase means the solution of a nonlinear equation system.

Let consider the following shallow space truss dome (see in Figure 5.1) where the computation of critical points could be very crucial point of the instability investigation.

The theory and the results of the equilibrium curve investigation - obtained using the *higher-order predictor-corrector algorithm* - are presented in (Csébfalvi, 1998).

In order to compare the results of the *linear*, *quadratic* and *cubic* approximation are presented in Figure 5.2-5.4. The estimated equilibrium path points are marked by dots in the curve. In each case, the number of iterations was 50. It is worth mentioning what the total number of corrector iteration steps were in each case.

The benefit of the method proposed is confirmed by the nonlinear stability analysis of beams and snap-through analysis of medium size three-dimensional truss and frames. The suggested numerical method can be easily incorporated into a computer program for nonlinear finite element analysis, and it is believed to enhance the accuracy of the program. The success of the method proposed for large-scale problems still needs verification.

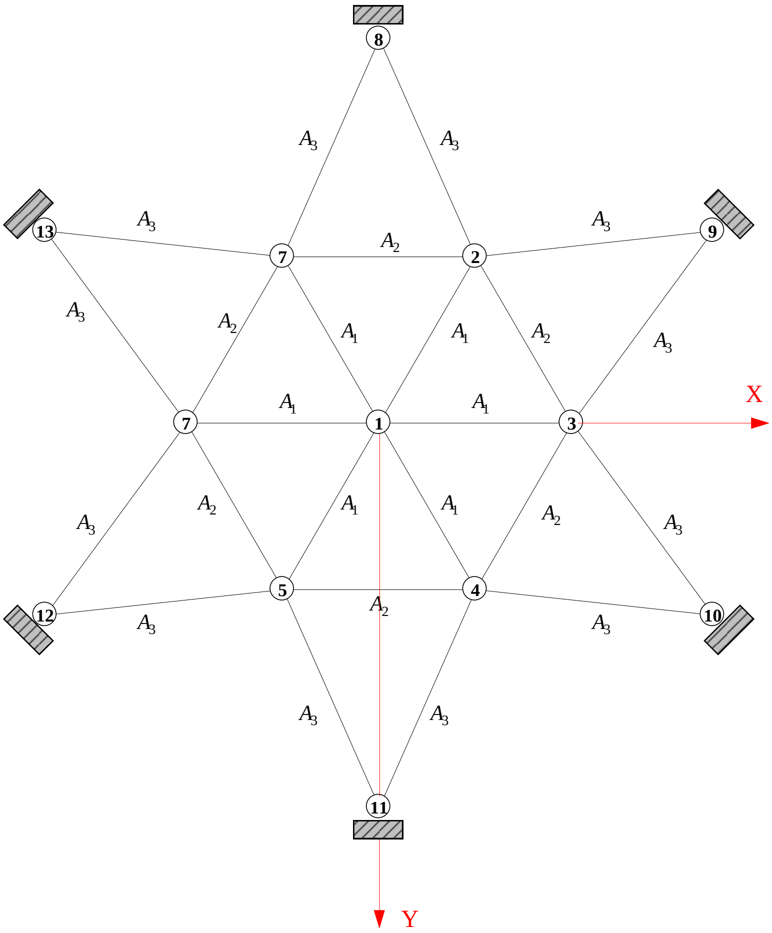

Figure 5.1: The shallow truss-dome problem

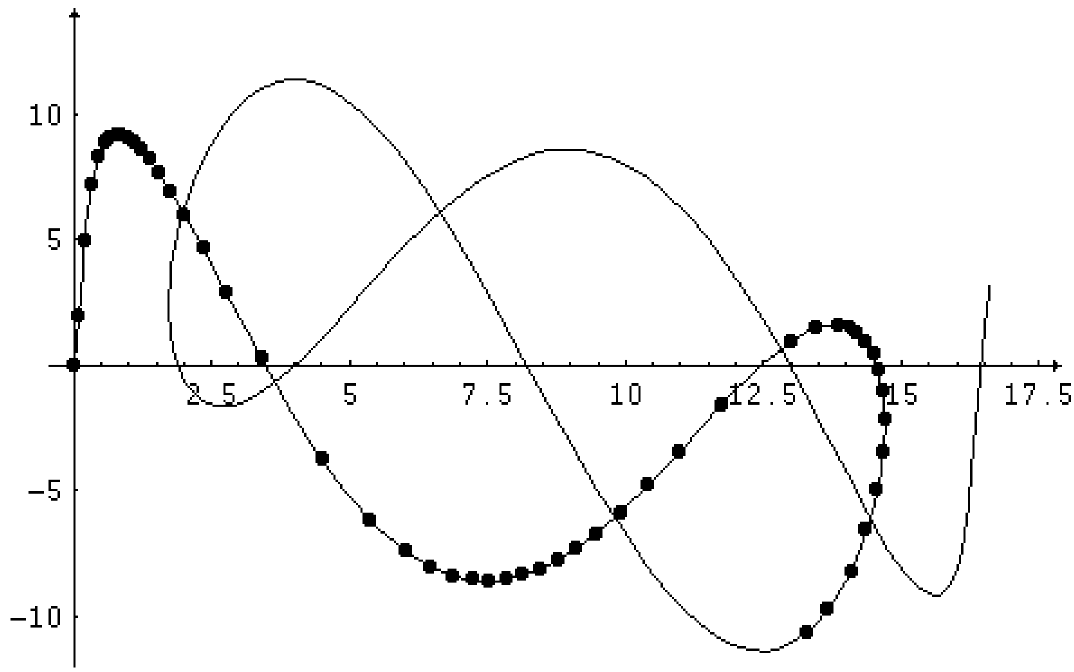

Figure 5.2: The equilibrium path of shallow truss-dome problem – first order approximation

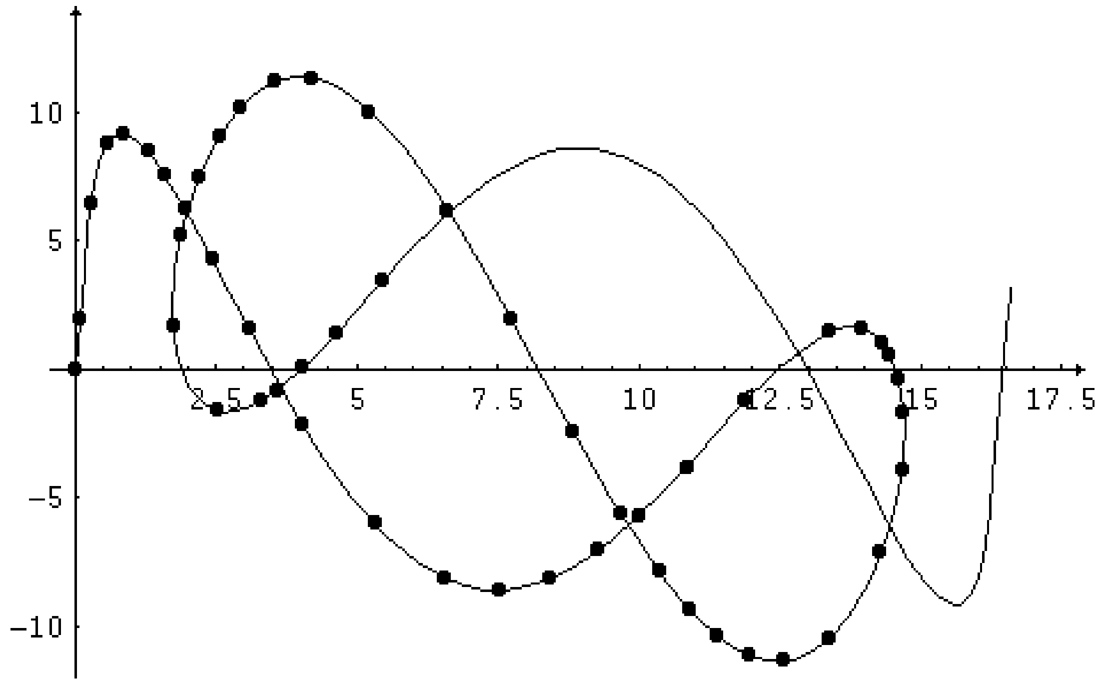

Figure 5.3: The equilibrium path of shallow truss-dome problem – second order approximation

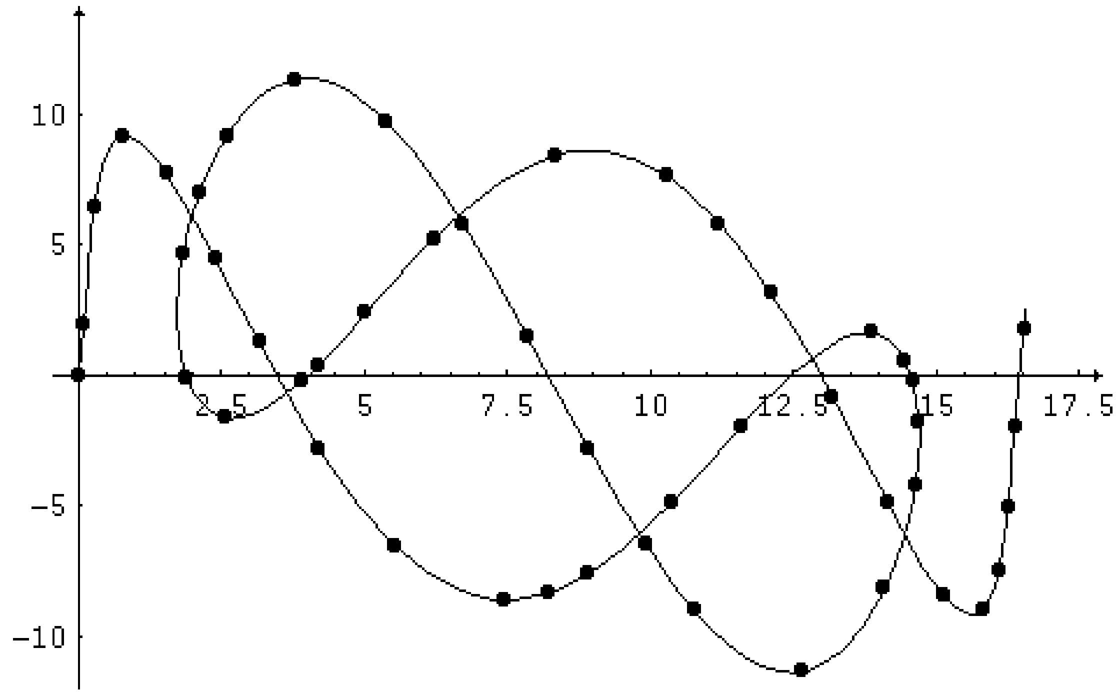

Figure 5.4: The equilibrium path of shallow truss-dome problem – third order approximation

The total potential energy contents two parts, the internal energy (strain energy) and external energy functions in terms of the vector of design variables and state variables namely here the vector of nodal displacements.

$$V\left(\boldsymbol{X},\boldsymbol{D}\right)=U\left(\boldsymbol{X},\boldsymbol{D}\right)-\boldsymbol{D}^{T}\boldsymbol{F} \tag{5.4}$$

where $U\left(\boldsymbol{X},\boldsymbol{D}\right)$ is the strain energy function in terms of the generalized vector of displacements $\boldsymbol{D}\in\Re^{N}$. $\boldsymbol{F}\in\Re^{N}$ is the vector of external loads, where N is the number of freedom.

The equilibrium equations are given from the principle of stationary value of total potential energy in terms of the vector of design variables and state variables namely here the vector of nodal displacements:

$$\frac{\partial V\left(\boldsymbol{X},\boldsymbol{D}\right)}{\partial\boldsymbol{D}}=\frac{\partial U\left(\boldsymbol{X},\boldsymbol{D}\right)}{\partial\boldsymbol{D}}-\boldsymbol{F}=0 \tag{5.5}$$

where the load is given by the multiplication of a basic value and a load intensity factor.

The number of equilibrium equations will be exactly same as the number of freedom, e.g. the number of elements of nodal displacement vector. The second part of the equilibrium equations contents only the external loads. Therefore, in the following formulas only the strain energy formulation will be considered.

The strain energy of one element – supposed linear elastic material law - is formulated as follows.

$$U_e=\frac{1}{2}E_eA_e\int_0^{L_e}\varepsilon^2d\xi=\frac{1}{2}\int_0^{L_e}\frac{\left(\varDelta L_e\right)^2}{L_e^2}d\xi=\frac{1}{2}\frac{E_eA_e}{L_e}\left(\varDelta L_e\right)^2 \tag{5.6}$$

where E_eA_e $\left(e=1,2,...,M\right)$ is the rod stiffness, E_e is the elasticity modulus, A_e is the cross-section of the elements, M the number of elements. L_e is the length of the elements, and $\varDelta L_e$ is the shortening or elongation of the elements depending on the sign of the internal forces.

The shortening or elongation of the elements will be computed in terms of the original coordinates of the geometry and the nodal displacements.

$$\varDelta L_e=\sqrt{\left(\varDelta x_e+\varDelta u_{e,x}\right)+\left(\varDelta y_e+\varDelta u_{e,y}\right)+\left(\varDelta z_e+\varDelta u_{e,z}\right)}-\sqrt{\varDelta x_e^2+\varDelta y_e^2+\varDelta z_e^2} \tag{5.7}$$

where Δx_e, Δy_e, Δz_e are the components of the element in x, y, z coordinate system. The displacements ($\Delta u_{e,d} = u_{2,d} - u_{1,d}$), $(d = x, y, z)$ of the element is computed in the 3D space.

Substituting into the strain energy function

$$U_e = \frac{1}{2}\frac{E_e A_e}{L_e}\left(\sqrt{(\Delta x_e + \Delta u_{e,x}) + (\Delta y_e + \Delta u_{e,y}) + (\Delta z_e + \Delta u_{e,z})} - \sqrt{\Delta x_e^2 + \Delta y_e^2 + \Delta z_e^2}\right)^2 \qquad (5.8)$$

The equilibrium equations are given by differentiation in terms of nodal displacements

$$\frac{\partial}{\partial u_j}\left(\sum \frac{1}{2}\frac{E_e A_e}{L_e}\left(\sqrt{(\Delta x_e + \Delta u_{e,x}) + (\Delta y_e + \Delta u_{e,y}) + (\Delta z_e + \Delta u_{e,z})} - \sqrt{\Delta x_e^2 + \Delta y_e^2 + \Delta z_e^2}\right)\right) = \mathrm{F} \qquad (5.9)$$

After the solution of (5.9) equilibrium equation system the nodal displacements resulted directly. Substituting the obtained displacements into the equilibrium equation system the stress constraints and displacements already could be formulated as follows:

$$0 \le \sigma_e \le \sigma^U \qquad (5.10)$$

$$\sigma_e^L \le -\sigma_e \le 0 \qquad (5.11)$$

$$u_k^L \le u_k \le u_k^U, \quad (k = 1,2,\ldots,N) \qquad (5.12)$$

We have to note that in this case the constraints are given always in implicit form because the displacements depend on the actual value of the cross sectional areas of the structural elements. Therefore, the optimization problem could be solved using only heuristic or metaheuristic methods.

If the local stability consideration is required as well than we need further criteria for Euler buckling mode or any others when we have e.g. thin-walled structures:

$$\sigma_e^L = max\left\{\sigma^L,\ \sigma_e^E, \sigma_e^B\right\}, \qquad (5.13)$$

where σ_e^E is the Euler buckling constraints, and σ_e^B is given by the design code.

5.2 Nonlinear Modelling of Three-Bar Truss

In this chapter the *geometrically nonlinear structural modelling* is presented and the computation of the structural constraints and the mathematical formulation will be defined for minimal weight design subject to stress and displacements constraints of a simple three-bar truss (see in Figure 5.5).

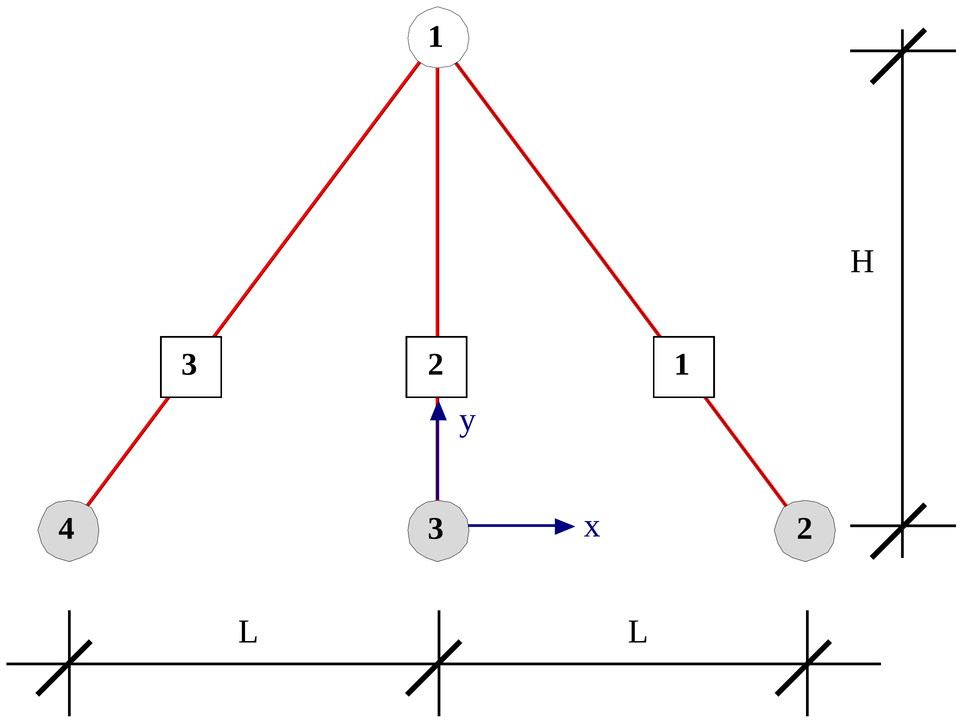

Figure 5.5: The initial data of three-bar truss nonlinear problem

The structure is given in xy coordinate system. The nodes and elements are numbering as shown in Figure 5.5. The loads are acting on the free join (1) and its direction is given in xy coordinate system. The computation is based on the nonlinear equilibrium equation system in terms of nodal displacements as defined in chapter 5.1.

Previously we stated that in case of nonlinear modelling the *structural constraints* will be *implicit functions* of *design variables* and *state variables* as well. Consequently, only *heuristic solution methods* can be applied where an initial design is supposed. The initial design is *randomly selected* from the range of cross-sectional areas. Starting from the total potential energy function differentiate by the nodal displacements, we get the structural constraints.

The computation formula of a deformed element is given by the subroutine of Table 5.1-Table 5.2. The steps of the modelling and computation is given in Table 5.3-Table 5.9.

```
LoadedTruss: = Module[{i, left, right, x1, y1, x2, y2},

deformedtruss = {};


Do[

left = element[i, 1]; right = element[i, 2];

x1 = deformedpoint[left, 1]; y1 = deformedpoint[left, 2];

x2 = deformedpoint[right, 1]; y2 = deformedpoint[right, 2];

AppendTo[deformedtruss, Graphics[{RGBColor[1,0,0], Line[{{x1, y1}, {x2, y2}}]}]];

, {i, 1, e}];

Return[deformedtruss];

];


PlaneTruss: = Module[{i, left, right, x1, y1, x2, y2, Radius},

radius = 0.5;

startingtruss = {};


Do[left = element[i, 1]; right = element[i, 2];


x1 = startingpoint[left, 1]; y1 = startingpoint[left, 2];

x2 = startingpoint[right, 1]; y2 = startingpoint[right, 2];


AppendTo[startingtruss,


Graphics[{Thickness[y[i]/H], RGBColor[0,0,0], Line[{{x1, y1}, {x2, y2}}]}]];

AppendTo[startingtruss,
```

Table 5.1: Nonlinear computation of trusses – Part 1

```
Graphics[{EdgeForm[Thin], White, Rectangle[{(2 * x1 + 4 * x2)/6 − radius/2,
(2 * y1 + 4 * y2)/6 − radius/2}, {(2 * x1 + 4 * x2)/6 + radius/2,
(2 * y1 + 4 * y2)/6 + radius/2}]}]];

AppendTo[startingtruss,

Graphics[{RGBColor[0,0,0], Text[StyleForm[i, FontSize−> 10,
FontWeight−> "Bold"], {(2 * x1 + 4 * x2)/6 , (2 * y1 + 4 * y2)/6}]}]];,
{i, 1, e}];

Do[x1 = startingpoint[i, 1]; y1 = startingpoint[i, 2];
If[i == 1,

AppendTo[startingtruss, Graphics[{White, Disk[{x1, y1}, radius/2]}]],
AppendTo[startingtruss,

Graphics[{LightGray, Disk[{x1, y1}, radius/2]}]]];
AppendTo[startingtruss,

Graphics[{Thin, RGBColor[0,0,0], Circle[{x1, y1}, radius/2]}]];

AppendTo[startingtruss, Graphics[{RGBColor[0,0,0], Text[StyleForm
[i, FontSize−> 10, FontWeight−> "Bold"], {x1, y1}]}]];
, {i, 1, p}];

Return[startingtruss]; ];
```

Table 5.2: Nonlinear computation of trusses – Part 2

```
crlf = "\n";
Echo = False;
p = 4;
e = 3;
d = 2;
n = 2;
Clear[x, y, L, R, H, L, P, α, LO, LD];
points = Array[point, {p, d}]; startingpoints = Array[startingpoint, {p, d}];
H = 4;
L = 3;
point[1,1] = 0;
point[1,2] = H;
point[2,1] = +L;
point[2,2] = 0;
point[3,1] = 0;
point[3,2] = 0;
point[4,1] = −L;
point[4,2] = 0;


Do[Do[startingpoint[i, j] = point[i, j], {j, 1, d}], {i, 1, p}];
elements = Array[element, {e, 2}];
element[1,1] = 1;
element[1,2] = 2;
element[2,1] = 1;
element[2,2] = 3;
element[3,1] = 1;
element[3,2] = 4;
```

Table 5.3: Initial data of three-bar truss – Part 1

```
Clear[y];

Table[y[i], {i, e}];

Clear[x];

Table[x[i], {i, n}];

Clear[s];

Table[s[i], {i, e}];

Clear[load];

Table[load[i], {i, n}];

Clear[LO, LD];

Do[

bal = element[ea, 1]; job = element[ea, 2];

LO[ea] = 0;

Do[LO[ea] = LO[ea] + (point[bal, j] − point[job, j])^2, {j, 1, d}];

LO[ea] = Sqrt[LO[ea]];

If[Echo, Print["LO[", ea, "]  = ", LO[ea]]];

, {ea, 1, e}];

point[1,1] = point[1,1] − x[1];

point[1,2] = point[1,2] − x[2];

Do[Do[deformedpoint[i, j] = point[i, j], {j, 1, d}], {i, 1, p}];

S = {};

Do[

bal = element[i, 1]; job = element[i, 2];

LD[i] = 0;

Do[LD[i] = LD[i] + (point[bal, j] − point[job, j])^2, {j, 1, d}];

LD[i] = Sqrt[LD[i]];

s[i] = (LD[i] − LO[i])/LO[i] ∗ modulus; AppendTo[S, s[i]];

If[Echo, Print["LD[", ea, "]  = ", LD[ea]]];

, {i, 1, e}];
```

Table 5.4: Initial data of three-bar truss – Part 2

```
If[Echo, Print["S  = ", S]];

Clear[W, OWF, PEF];

W = y[1] * LO[1] + y[2] * LD[2] + y[3] * LD[3];

OWF = x[1] * load[1] + x[2] * load[2];

PEF = 0; Do[PEF = PEF + y[i] * (LD[i] − LO[i])^2/LO[i] , {i, e}];

PEF = modulus * PEF/2 − OWF;

If[Echo, Print["PEF(BEF)  = ", PEF]];

PEF = Simplify[PEF];

If[Echo, Print["PEF(AFT)  = ", PEF]];

PEF

PX = 600;

PY = 1200;

load[1] = PX;

load[2] = PY;

modulus = 200 * 10^6;

density = 1;

y[1] = 0.004; y[2] = 0.003; y[3] = 0.004;

Y = {}; Do[AppendTo[Y, y[i]], {i, 1, e}];

plot1 = Show[PlaneTruss, ImageSize → 300]; Print[plot1];
```

Table 5.5: Computation of three-bar truss – Part 1

```
Clear[eqn, eqnm, jac, jacm];

Quet[

Do[y[i] =., {i, 1, e}];

];


eqnm = Array[eqn, {n}]; jacm = Array[jac, {n, n}];
```

Table 5.6: Computation of three-bar truss – Part 2

```
Do[eqn[i] = D[PEF, x[i]], {i, n}];

If[Echo, Print["eqnm  = ", MatrixForm[eqnm]]];

If[Echo, Print["S  = ", MatrixForm[S]]];

Do[jac[i, j] = D[eqn[i], x[j]], {i, n}, {j, n}];

If[Echo, Print["jacm  = ", MatrixForm[jacm]]];

Clear[dedy, dedym];

dedym = Array[dedy, {n, e}];

Do[dedy[i, j] = D[eqn[i], y[j]], {i, n}, {j, e}];

If[Echo, Print["dedym  = ", MatrixForm[dedym]]];


S = {};

Do[s[i] = (LD[i] − LO[i])/LO[i] ∗ modulus; AppendTo[S, s[i]], {i, e}];

If[Echo, Print["S  = ", MatrixForm[S]]];

Clear[dsdx, dsdxm];

dsdxm = Array[dsdx, {e, n}];

Do[dsdx[j, i] = D[s[j], x[i]], {j, e}, {i, n}];

If[Echo, Print["dsdxm  = ", MatrixForm[dsdxm]]];

If[Echo, Print["OWF(C)  = ", OWF]];

Quiet[Do[y[i] =., {i, 1, e}]; Do[x[i] =., {i, 1, n}]];


FixedAngle = True;

goal = 0; Do[goal = goal + LO[i] ∗ y[i], {i, 1, e}];

rels = {}; Do[AppendTo[rels, Quiet[eqn[i] == 0, Set::write]], {i, 1, n}];

vars = {};

cony = {y[1] == 0.004, y[2] == 0.003, y[3] == 0.004};

Do[AppendTo[vars, y[i]], {i, 1, e}];

Do[AppendTo[vars, x[i]], {i, 1, n}];
```

Table 5.7: Computation of three-bar truss – Part 3

```
ExactSolution = NMinimize[{goal, {rels, cony}}, vars,

Method → {"RandomSearch", Method → "InteriorPoint", "SearchPoints" → 10,

Print["ExactSolution  = ", Chop[ExactSolution, 0.0001]];


X = {}; Do[x[i] = Replace[x[i], ExactSolution[[2]]]; AppendTo[X, x[i]], {i, 1, n}];

Y = {}; Do[y[i] = Replace[y[i], ExactSolution[[2]]]; AppendTo[Y, y[i]], {i, 1, e}];


{X, Y, S} = Chop[{X, Y, S},0.0001];

Print["Y  = ", PaddedForm[Y, {8,4}]];

Print["X  = ", PaddedForm[X, {8,4}]];

Print["S  = ", PaddedForm[S, {8,4}]];


plot2 = Show[{LoadedTruss, PlaneTruss}, ImageSize → 300]; Print[plot2];
```

Table 5.8: Computation of three-bar truss – Part 4

```
"ExactSolution  = "{0.052000000000000005, {y[1] → 0.004, y[2] → 0.003, y[3] → 0.004,

x[1] → 0.005213447826819619, x[2] → 0.0033843597515275537}}


"Y  = "{"0.0040", "0.0030", "0.0040"}

"X  = "{"0.0052", "0.0034"}X  =

"S  = "{"16977.0510", " − 169047.9700", " − 233403.9200"}S  =
```

Table 5.9: Results of nonlinear three-bar truss optimization problem

5.3 Nonlinear Modelling of 24-Bar Dome Truss

The 24-bar truss behavior has been already presented in chapter 5.1. In this session the computational details will be discuss. The geometry is given in Figure 5.6 and in Table 5.10-Table 5.11.

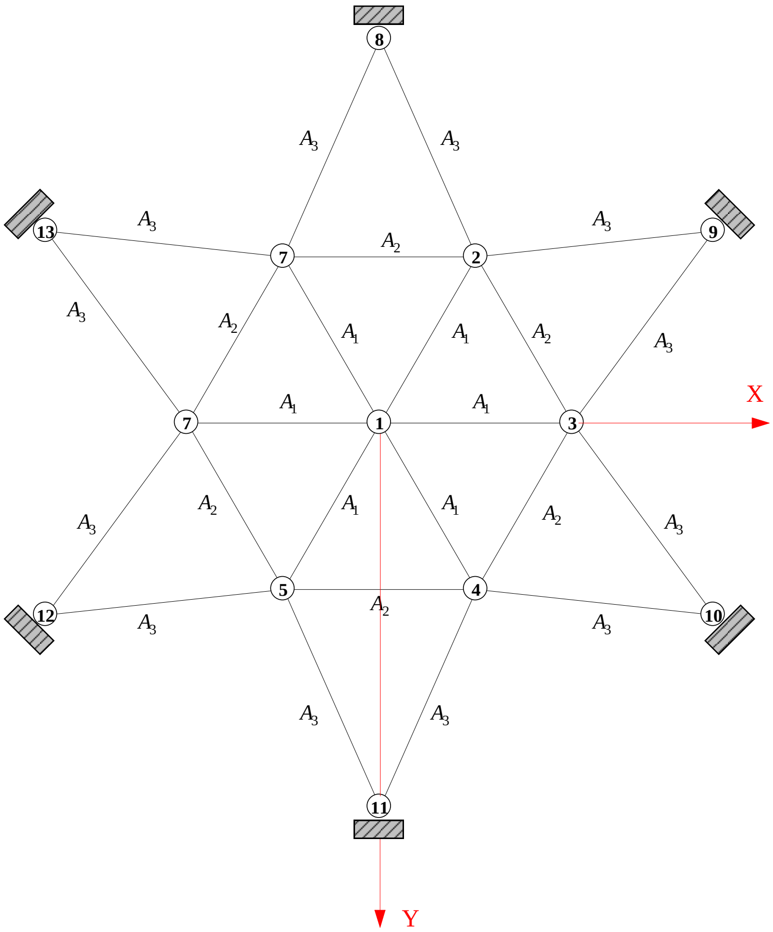

Figure 5.6: The joints and elements of the shallow truss-dome structure

The example is based on the results of (Criesfield, 1981) obtained four different critical points on the primary equilibrium path if the following load values are considered: 0.5 at the central node and 1.0 unit at nodes 2-7.

The first singular point is a single bifurcation ($\lambda_1 = 8.68$), while the following two are double bifurcation points ($\lambda_2 = 10.26$, and $\lambda_3 = 15.67$). Only the fourth singular point is a simple limit point ($\lambda_4 = 18.40$) that confirms the hazardous of the theories and methods which are able to tackle only snap-through phenomenon.

In this paper, a weight optimization is considered subjected to global stability constraints. The cross-sections as design variables are involved into three groups (Figure 5.6). The load intensity factor is changing from zero to one.

Nodes	X [cm]	Y [cm]	Z [cm]
1	0	0	8.216
2	12.50	21.65063509	6.2.16
3	25.00	0	6.216
8	0	50.00	0
9	43.330127019	25.00	0

Table 5.10: Initial coordinates of 24-bar shallow space truss

Design variables	$A_i \in [1.00;\ 2.00]\ (cm^2);\ i \in \{1,2,3\}$	
Load cases	Nodes	Z
1	1	$-5.00\ kN$
	2, 3, 4, 5, 6, 7	$-10.00\ kN$
Material properties	Modulus of elasticity	$E = 10000\ \ kN/cm^2$

Table 5.11: Initial data of 24-bar shallow space truss

Using the proposed hybrid metaheuristic method, where the number of generations is 10 and the population size is 100, two optimization problems are considered.

Case 1:

In first case, a sizing optimization problem is solved for minimal volume optimization subjected to structural stability. The structure is loaded up to the maximal load intensity factor while the smallest eigenvalue becomes zero. The obtained best solution for the grouped design variables are the following: $A_1 = 1.000$; $A_2 = 1.321$; $A_3 = 1.119$. The optimal volume in this case is $V_{opt} = 773.127$.

Case 2:

In the second case, a sizing-shaping optimization problem is presented. The three sizing variables are extended with three shift variables namely the vertical position of all free joints ($Z_i; i = 1,2,\ldots,7$), and the horizontal position of the joints 2-7 ($R_j; j = 2,\ldots,7$). In this case, the same proposed hybrid metaheuristic method has been applied, with the number of generations 10 and the population size 100. The obtained best solution is the following: $A_1 = 1.000$; $A_2 = 1.378$; $A_3 = 1.084$; $Z_1 = 7.685$; $Z_{2-7} = 6.121$; $R_{2-7} = 24.665$. The optimal volume is $V_{opt} = 765.699$ and the lowest eigenvalue is zero for three digits in the best solution.

6. UNCERTAINTIES IN STRUCTURAL OPTIMIZATION

In the real-world structural optimization problems, the optimal performance obtained using conventional deterministic methods can be dramatically degraded in the presence of sources of uncertainty. The source of uncertainty may be the variability of applied loads, spatial positions of nodes, material properties, and so on. Various approaches have been developed to account for different types of uncertainty in structural design and optimization methods. Generally speaking, these methods are mainly based on two kinds of uncertainty models: *probabilistic* (stochastic) or *possibilistic* (fuzzy) models.

Based on *stochastic uncertainty models* of mechanical parameters, various techniques were proposed by ((Marti, 1999), (Rozvany, 2001), (Choi, 2001), (Lógó, 2007)) for evaluation and estimation of failure probabilities that can be utilized in the reliability-based structural design methods.

Based on the *probability distribution of the random data*, and using decision theoretical concepts, optimization problems under stochastic uncertainty are converted into appropriate deterministic substitute problems. Due to the occurring probabilities and expectations, approximate solution techniques must be applied. Several deterministic and stochastic approximation methods are provided by (Marti, 2005). A fuzzy optimization approach for geometrical nonlinear space trusses was presented by (Kelesoglu, 2005). Assuming uncertain-but-bounded parameters (Ben-Haim, 1990) developed the so-called convex model, with which (Pantelides, 1989) proposed a robust truss optimization method. For various classes of convex optimization problems, a unified methodology of robust optimization was developed by (Ben-Tal, 2002). Calafiore (Calafiore, 2004) proposed a method for finding the ellipsoidal bounds of the solution set of uncertain linear equations by using the semidefinite program (SDP), which was presented originally by (Wolkowicz, 2000). We have to mention the pioneer works of (Achtziger, 1992) and (Achtziger, 1997) as well, related to truss topology design and topology optimization of discrete structures, which contain the basic theoretical backgrounds on the field of the truss topology optimization.

The *multiple-load truss topology and sizing optimization* was presented first time by (Achtziger, 1998). Similar work was presented by (Alvarez, 2005) for minimization of the expected compliance as an alternative approach for multiple load truss optimization, which seems also very useful to tackle the uncertain optimization problems. In a recent paper, (Dunning, 2011) introduced a new probabilistic approach for robust topology optimization to minimize the volume-constrained expected compliance with uncertainty in loading magnitude and applied direction, where uncertainties are assumed normally distributed and statistically independent. The presented model was formulated as a statistical model, which after some manipulation was replaced by an equivalent multiple load problem in the function of the number of perturbed loads.

6.1 Load Direction Uncertainty of Trusses

In this chapter, using a recently developed unified approach (Csébfalvi, 2018) we present benchmark results for structural optimization when the only source of uncertainty is the variability of the applied load directions. The novel worst-direction-oriented unified approach can be applied to a broad class of engineering optimization problems. In each case, the central element of the solution searching algorithm is a standard deterministic multi-load structure optimization problem, which using an appropriate method, can be solved within reasonable time. The essence of the novel conception is independent from the theoretical description of the directional uncertainty, which may be either *probabilistic* (*stochastic*) or *possibilistic* (*fuzzy*).

In the presented *unified approach* (non-probabilistic and non-possibilistic), the varying load directions are handled by quadratic and linear constrains, which describe spherical regions around the nominal loads. Naturally the applied load direction handling method can be replaced by any other uncertainty representation form, which can be described by an appropriate combination of linear or quadratic (linearizable) relations.

The result of the optimization is a structure design with minimum performance measure which is invariant to the investigated uncertainty type and satisfies the constraints with a given constraint tolerance. In order to demonstrate the viability, variability and efficiency of the proposed new approach, we present problem-specific models and algorithms with detailed and well-illustrated benchmark results for topology optimization of 2D continuum structures and cross-section size optimization of 2D truss structures with displacement and stress constraints. It will be demonstrated that in each case the computational cost of the new approach is comparable with its deterministic equivalent because its central element is a problem-dependent deterministic multi-load structure optimization problem and the problem-dependent worst-load-direction searching algorithm can be formulated as a much smaller linearly and quadratically constrained quadratic or a quadratically constrained linear programming problem which can be solved by several ways efficiently.

Generally, the standard deterministic truss weigh minimization problem with cross-sectional areas of the bars as continuous design variables and nodal point displacements and element stresses as response variables can be written as follows:

$$\boldsymbol{w}(\boldsymbol{a}) \rightarrow \boldsymbol{min} \tag{6.1}$$

$$\underline{\boldsymbol{a}} \leq \boldsymbol{a} \leq \overline{\boldsymbol{a}} \tag{6.2}$$

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}(\boldsymbol{a}, \boldsymbol{p}) \leq \overline{\boldsymbol{u}} \tag{6.3}$$

$$\underline{\boldsymbol{\sigma}} \leq \boldsymbol{\sigma}(\boldsymbol{a}, \boldsymbol{p}) \leq \overline{\boldsymbol{\sigma}} \tag{6.4}$$

$$\boldsymbol{e}(\boldsymbol{a}, \boldsymbol{p}) = \boldsymbol{0} \tag{6.5}$$

where n is the number of bars, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ is the vector of the design variables, $\boldsymbol{w}(\boldsymbol{a})$ is the weight of the structure, m is the total number of displacements (degrees of freedom), $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)$ is the vector of applied loads, $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$are the implicit response variables (nodal point displacements and element stresses). The vector of equilibrium equations is denoted by $\boldsymbol{e}(\boldsymbol{a}, \boldsymbol{p}) = (e_1(\boldsymbol{a}, \boldsymbol{p}), e_2(\boldsymbol{a}, \boldsymbol{p}), \ldots, e_m(\boldsymbol{a}, \boldsymbol{p}))$. In the formulations, the under-bar (upper-bar) symbol always means lower (upper) bound. In the case of plane trusses, each applied load can be described in the following form: $p_i = \{ px_i, py_i \}$, $i \in \{1, 2, \ldots, m\}$, where px_i (py_i) means the horizontal x-direction (vertical y-direction) load components.

In the traditional deterministic formulation $\boldsymbol{p}$ is a constant vector (the loads are nominal loads). In this paper, we considered the following uncertainty on the loading: In addition to the nominal loading, each node is subjected to a load having random direction and constant intensity. The additional uncertain load vector $\boldsymbol{r}$, where $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ can be described in the following parametric form:

$$r_i = R_i * \left\{ \{ rx_i, ry_i \} \mid rx_i^2 + ry_i^2 = 1, \; i \in \{1, 2, \ldots, m\} \right\} \tag{6.6}$$

where R_i is the load intensity.

According to the modification, the goal is to minimize the weight of the truss on the set of the feasible $\boldsymbol{d} = \boldsymbol{p} + \boldsymbol{r}$ load combinations. In the following let $\boldsymbol{D}$ denote the feasible load combinations. It is worth noting that after adding $\boldsymbol{r}$ to $\boldsymbol{p}$ we get a mathematical program with infinite number of constraints since because each quadratic equation define a continuous set with uncountable infinite number of elements. The result of the optimization will be a robust minimal-weight truss design, which is invariant to the investigated load uncertainty type.

6.2 Weight Minimization of Trusses with Varying Load Directions

In order to avoid dealing with infinite number of constraints we have to replace the original problem with a more tractable equivalent algorithm based on a finite number of constraint sets.

The essence of the presented algorithm is very simple. Fixing the design variables according to the optimal nominal solution we solve step by step $n+m$ minimization and $n+m$ maximization problems on set of the feasible loads selecting exactly one response variable as an objective function in each step.

If the $2\times(n+m)$ dimensional pareto-front is feasible then the robust algorithm terminates and the nominal solution will be the optimal robust solution.

Otherwise we append the constraint sets of the detected unfeasible loads to the constraint set of the nominal load, solve the weight minimization problem on the extended constraint set and repeat the pareto-front investigation.

According to the verbal description of the essence of the algorithm the mathematical formulation of the generic step may be the following:

Let $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_{s(i)}\}$ the set of the load sets at the start of step i, where $i \in \{1, 2, \ldots\}$. In this case we have to solve the following weight minimization problem for $\boldsymbol{a}$:

$$\boldsymbol{w}(\boldsymbol{a}) \rightarrow \boldsymbol{min} \tag{6.7}$$

$$\underline{\boldsymbol{a}} \leq \boldsymbol{a} \leq \overline{\boldsymbol{a}} \tag{6.8}$$

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}(\boldsymbol{a}, \boldsymbol{d}_j) \leq \overline{\boldsymbol{u}}, \quad j \in \{1, 2, \ldots, s(i)\} \tag{6.9}$$

$$\underline{\boldsymbol{\sigma}} \le \boldsymbol{\sigma}\left(\boldsymbol{a}, \boldsymbol{d}_j\right) \le \overline{\boldsymbol{\sigma}}\ , \quad j \in \{1, 2, \ldots, s(i)\} \tag{6.10}$$

$$\boldsymbol{e}\left(\boldsymbol{a}, \boldsymbol{d}_j\right) = \boldsymbol{0}\ , \quad j \in \{1, 2, \ldots, s(i)\} \tag{6.11}$$

Having solution $\boldsymbol{a}$ we have to solve the following $n+m$ minimization and $n+m$ maximization problems on the set of the feasible loads:

$$o \to \boldsymbol{min}\left(\boldsymbol{max}\right) \tag{6.12}$$

$$o \in \boldsymbol{u} \cup \boldsymbol{\sigma} \tag{6.13}$$

$$\boldsymbol{e}\left(\boldsymbol{a}, \boldsymbol{d}_j\right) = \boldsymbol{0} \tag{6.14}$$

$$\boldsymbol{d} \in \boldsymbol{D} \tag{6.15}$$

After that the selection of the unfeasible loads (the sample update) is straightforward.

The feasibility of a load $\boldsymbol{d} \in \boldsymbol{D}$ for a given $o \in \boldsymbol{u} \cup \boldsymbol{\sigma}$ response variable value may be checked by the following non-smooth function of the "normalized" constraint violation terms:

$$\lambda = \boldsymbol{max}\left\{ \bar{o}, \vec{o} \right\} \tag{6.16}$$

where

$$\left\{ \bar{o}, \vec{o} \right\} = \left\{ \boldsymbol{max}\left(0, 100 * \frac{\underline{o} - o}{\underline{o}}\right), \boldsymbol{max}\left(0, 100 * \frac{o - \bar{o}}{\bar{o}}\right) \right\} \tag{6.17}$$

A pareto-front is feasible, if $\lambda \le \varepsilon$ where ε is the maximum allowable percentage constraint violation. In the present study, an extremely small $\varepsilon = 1\%$ tolerance was applied.

6.3 Robust Optimization of Ten-Bar Truss

In this section, we present detailed computational results for the well-known ten-bar truss weight minimization problem to illustrate the essence the presented new robust approach. According to the usual assumptions, the length of each vertical and horizontal bar is 360 in. The constraints involve the stress in each member and the displacements at the nodes. The elements (nodes) are numbered by consecutive positive integers. The design variables $0.1 \le a_i \le 35,\ i \in \{1, 2, \ldots, 10\}$ are the cross-sectional areas of the bars. The allowable element stresses $\sigma_i,\ i\in\{1,2,\dots,10\}$ are ± 50 ksi and the nodal displacements $u_i,\ i\in\{1,2,\dots,8\}$ are limited to ± 5 in the horizontal and vertical directions. The displacements are indexed by consecutive positive integers in increasing node-direction order. For example, u_1 and u_2 describe the horizontal and vertical displacements of node one. The density of the material is $\rho = 0.1$ lb/in^3 and the elasticity modulus is $E = 1.0\times10^4$ ksi.

A rightward horizontal load of 100 kips acts at node 1, a vertical downward loads of 100 kips acts at nodes 2, a vertical upward loads of 100 kips acts at nodes 3, and a leftward horizontal load of 100 kips acts at node 4. According to the definition, the nominal loading pattern can be described by the set of its $x-y$ coordinate pairs:

$$\boldsymbol{p} = 100 * \{\{1,0\},\{0,-1\},\{0,1\},\{-1,0\}\}.$$

In our benchmark problem each nominal load $p_i = \{px_i, py_i\},\ i\in\{1,2,\dots,4\}$ is perturbed by an additional load of 30 kips with a totally random direction $0\le\alpha_i\le 2\pi,\ i\in\{1,2,\dots,4\}$. The perturbation load set $\boldsymbol{r}$ can be described by the previously introduced quadratically constrained parametric form:

$$\boldsymbol{r} = 30 * \left\{\{rx_i, ry_i\}\ \middle|\ rx_i^2 + ry_i^2 = 1,\ i\in\{1,2,\dots,4\}\right\}.$$

We have to note, that the introduced loading pattern with different parameters and assumptions was previously investigated by several authors, for example, by (Ben-Tal, 2002) and (Calafiore, 2004). Unfortunately, due to the missing or misleading parameter descriptions, we were unable to reproduce correctly their results from the papers. Therefore, retaining the original loading pattern, we developed a similar but new example in a correctly and therefore reproducible form and presented its exact solution, which can be used for testing the quality of exact and heuristic solution procedures to be developed in the future.

We have to note, that the size of perturbing load is large enough and its directional variability is maximal, therefore this “overemphasized academic” example may be a challenging test problem for all possible computational approaches.

In Figure 6.1, we show the structure of our benchmark problem. In the applied visualization, the free (fixed) nodes are represented by white (grey) circles, the elements by white squares. The possible perturbation load directions are visualized by dotted circles. In this figure, the perturbation load directions have no real meanings because these are only illustrations.

In Figure 6.2 we show a feasible nominal design, which is probably one of the best solutions of the problem due to the applied "global" problem solving technique. In the graphical representation, each bar thickness is proportional to its cross-sectional area. The label of the figure presents the weight of the structure, the set of the cross-sectional areas, the set of the nodal-point displacements, and the set of the element stresses.

When we solved the problem, the Wolfram Mathematica 8.0 package was used, which is an excellent prototyping tool with a wide-range of state-of-the-art optimization solvers and other useful features. In the presented study, a very general and stable optimization tool, namely, the **`NMinimize`** solver was used with the following settings:

```
Method → {"RandomSearch", Method → "InteriorPoint","SearchPoints" → 10}.
```

According to our preliminary experiences, **`NMinimize`** with these settings behaves as a "global solver", which in a nonlinear developing environment is a very useful result. We note that setting: **`Method → "InteriorPoint"`** means a state-of-the-art nonlinear interior point solver.

Starting from the nominal solution the algorithm identified 19 feasible loads, for which at least one response variable is unfeasible on the set of feasible loads. In the pareto-front searching process also **`NMinimize`** was used with the previously presented robust and efficient settings. We have to emphasize that in its original form we have to solve a set of partly quadratically and partly linearly constrained optimization problems with a linear objective.

In this problem the quadratic constraints define the spherical regions around the nominal loads and the linear constraints define the equilibrium conditions for fixed design variables. Because the quadratic constraints can be approximated by a set of piecewisely linear constraints with arbitrary accuracy, therefore we get a standard linear programming problem, which using an interior point solver can be solved extremely quickly.

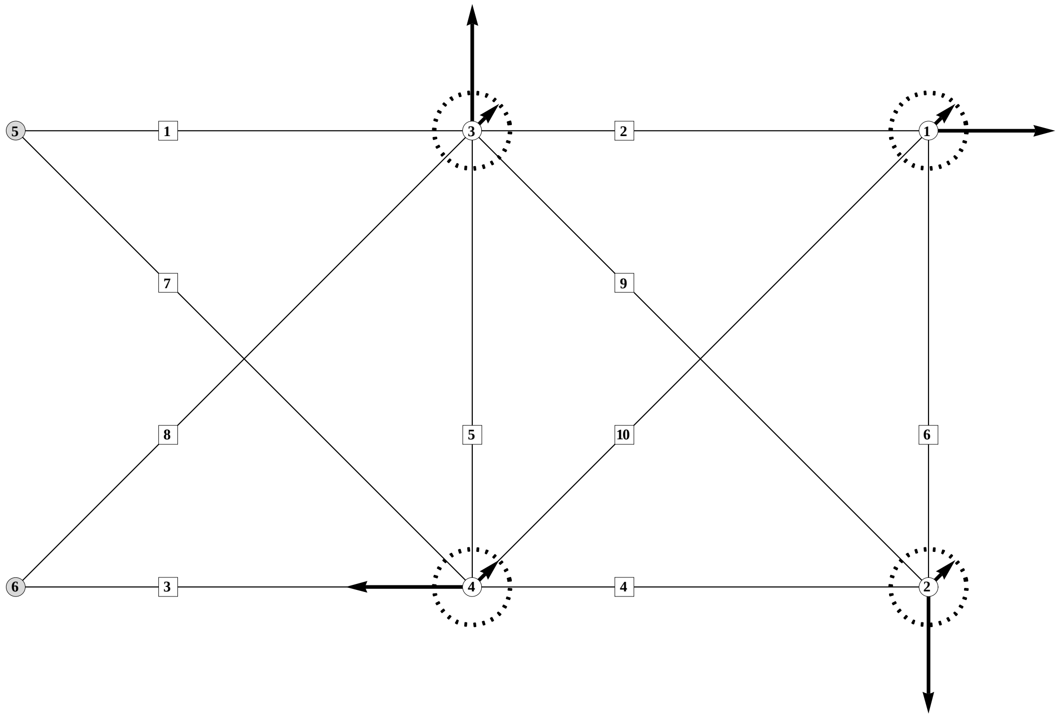

Figure 6.1: The structure and the loading pattern of the benchmark problem.

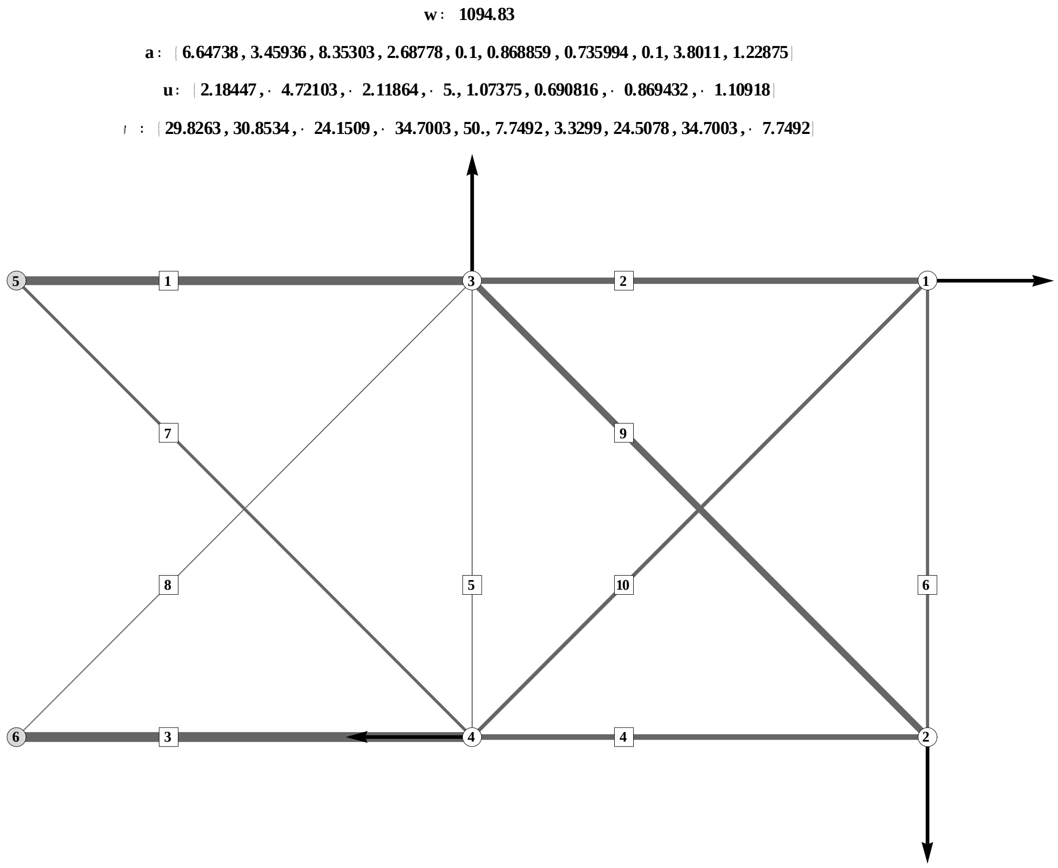

Figure 6.2: The best feasible nominal design.

After inserting the worst loads we get a multi-load problem with $1+19=20$ load cases. Solving this problem and checking the pareto-front for each response variable in each direction we get a solution, which is invariant to the allowed load perturbations. The fast convergence reveals that the presented robust approach is efficient, and theoretically and practically may be competitive with other previous approaches presented in literature.

We have to note, that (Calafiore, 2004) used a sample of 1,651 to get a probabilistically robust solution using a similar problem for the ten-bar problem. In Figure 6.3, we show the “worst-feasible-direction” from the identified 19 “bad-feasible-direction”, for which the relative percent error is $\varepsilon=503.951$ for the tension stress of element 8 on the set of the feasible load combinations using the nominal cross-sections in the pareto-front searching process:

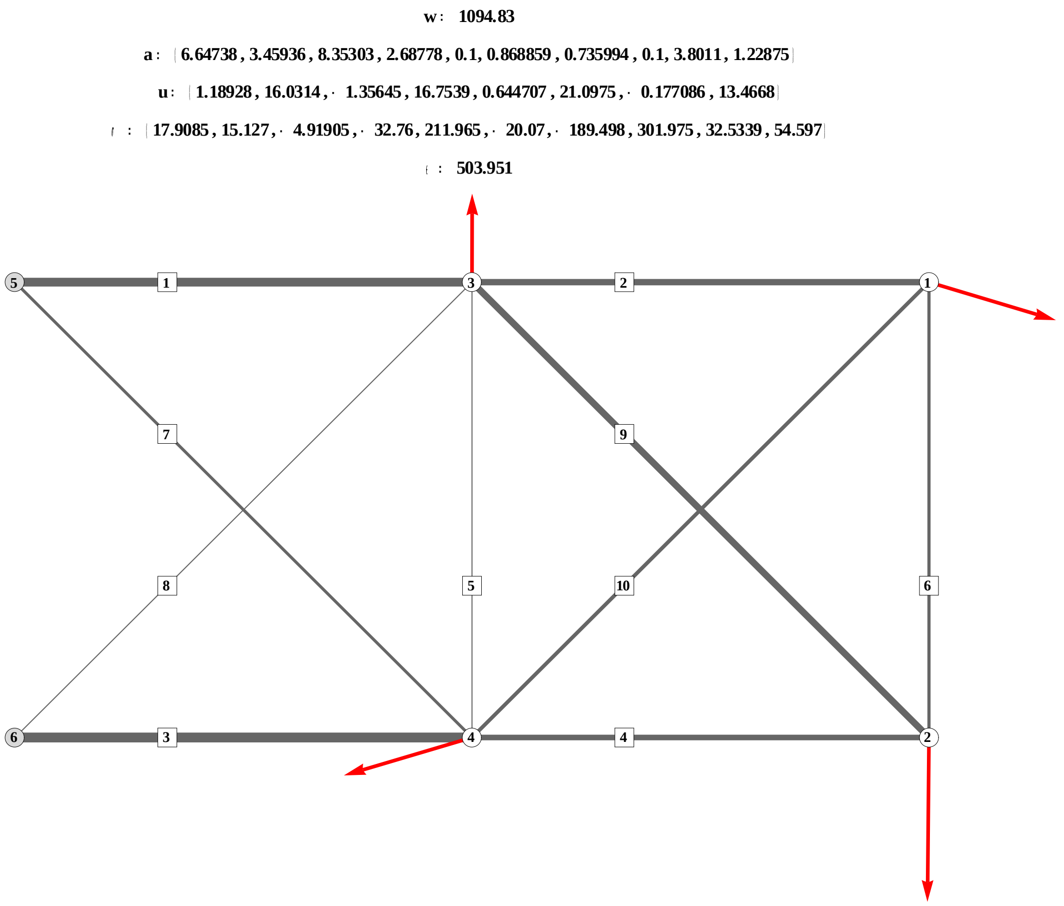

Figure 6.3: The worst-feasible-load-direction combination for $\boldsymbol{max}(\sigma_8)$

In Figure 6.4 we show the optimal solution, which we get in two steps with $\varepsilon=1\%$ relative percent error setting. We have to mention it again, that our solution is a really invariant to the uncertain loading directions within the given tolerance, which in other words, means that on the set of the directional load perturbations the probability of the failure is exactly zero.

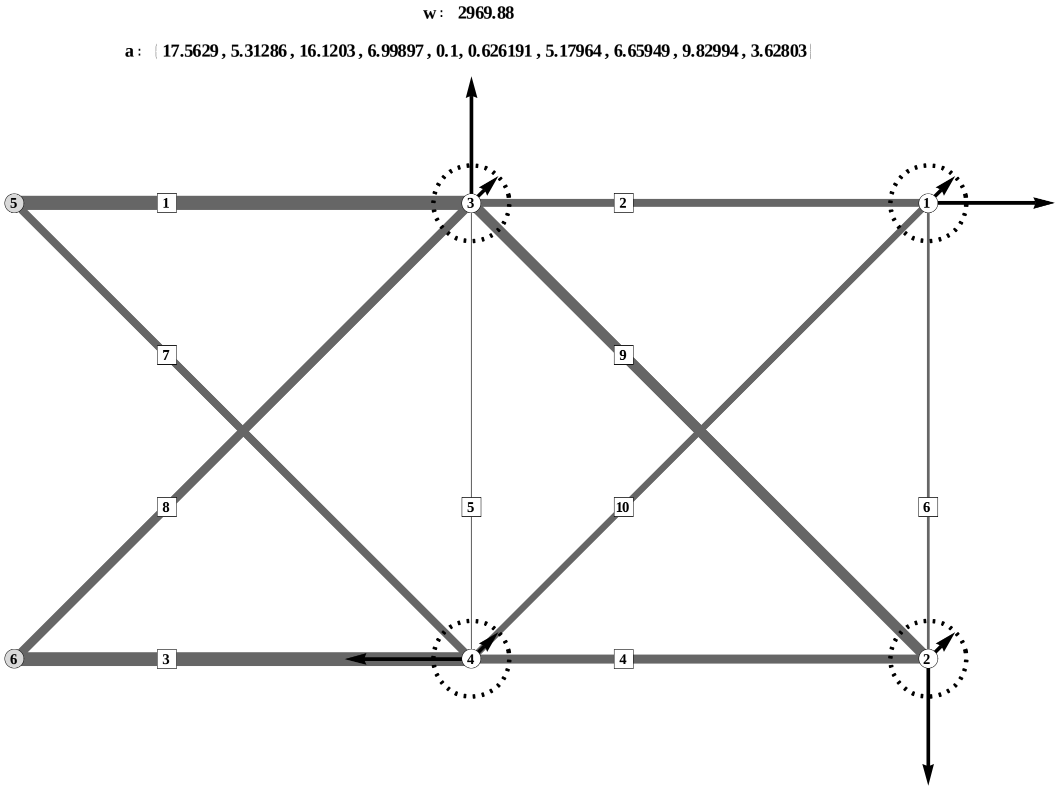

Figure 6.4: The best robust solution given by two steps.

In this context, failure can be defined in a very simple way: there is a feasible load combination, for which at least one response variable is unfeasible in at least one direction.

6.4 Volume Minimization with Varying Load Directions

Without loss of generality, we formulate our newly developed topology optimization model for continuum structures with uncertain load directions only for two-dimensional (2D) continuum problems. The standard topology optimization problem of continuum structures can be described as follows:

$$c(\boldsymbol{x}) = \boldsymbol{U}'\boldsymbol{K}\boldsymbol{U} \rightarrow \boldsymbol{min} \tag{6.18}$$

$$V(\boldsymbol{x}) = \varphi V_0 \tag{6.19}$$

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \tag{6.20}$$

$$\mathbf{0} \le \boldsymbol{x} \le \mathbf{1} \tag{6.21}$$

where c is the compliance, $\boldsymbol{U}$ and $\boldsymbol{F}$ are the global displacement and load vectors, respectively, $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{x}$ is the vector of design variables (the element densities), $V(\boldsymbol{x})$ and V_0 are the material volume and design domain volume, respectively, and φ is the prescribed volume fraction. The 2D design domain is assumed to be rectangular and discretized with square elements with four nodes per element and two degrees of freedoms (DOFs) per node. Both nodes and elements are numbered column-wise from left to right, and the DOFs $2n-1$ and $2n$ correspond to the horizontal and vertical displacement of node n, respectively. The optimization problem (1) can be solved by, for example, the well-known and the most widely used optimality criteria method or any other appropriate nonlinear solver (see, for example, (Liu, 2014)).

As it was demonstrated by (Andreassen, 2011), it is very easy to extend the algorithm to account for multiple load cases. In the case of m load cases, the load and displacement vectors can be defined as m column vectors and the objective function will be the sum of m compliances:

$$c(\boldsymbol{x}) = \sum_{i=1}^{m} \boldsymbol{U}_i' \boldsymbol{K} \boldsymbol{U}_i \rightarrow \boldsymbol{min} \tag{6.22}$$

$$V(\boldsymbol{x}) = \varphi V_0 \tag{6.23}$$

$$\boldsymbol{K}\boldsymbol{U}_i = \boldsymbol{F}_i, \quad i \in \{1, 2, \ldots, m\} \tag{6.24}$$

$$\mathbf{0} \le \boldsymbol{x} \le \mathbf{1} \tag{6.25}$$

Now, we will show that the multi-load compliance-minimization model, after simple modifications can be used to solve our directional uncertainty problem. Let $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{\alpha})$, where $\underline{\boldsymbol{\alpha}} \le \boldsymbol{\alpha} \le \overline{\boldsymbol{\alpha}}$, denotes a load vector with varying load directions. In this paper, we assume that each nominal load direction is an inner point of its angle set.

First, we rewrite the standard single load optimization model according to the varying load directions. The theoretical formulation of the modified optimization problem can be described as follows:

$$c(\boldsymbol{x}) = \boldsymbol{U}' \boldsymbol{K} \boldsymbol{U} \rightarrow \boldsymbol{min} \tag{6.26}$$

$$V(\boldsymbol{x}) = \varphi V_0 \tag{6.27}$$

$$\boldsymbol{KU} = \boldsymbol{F}(\boldsymbol{\alpha}) \tag{6.28}$$

$$\underline{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha} \leq \overline{\boldsymbol{\alpha}} \tag{6.29}$$

$$\boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{1} \tag{6.30}$$

According to the modification, the goal is to minimize the compliance of the structure for all the feasible loads $\boldsymbol{F}(\boldsymbol{\alpha})$, where $\underline{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha} \leq \overline{\boldsymbol{\alpha}}$. It is worth noting again, that after inserting the $\underline{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha} \leq \overline{\boldsymbol{\alpha}}$ relation we get a mathematical program with infinite number of constraints since set $\boldsymbol{\alpha}$ is in general a continuous set with uncountable infinite number of elements.

The result of the optimization will be a compliance-minimal design for the prescribed φ volume fraction, which is invariant to the investigated load uncertainty type. As we are only interested in linear elastic structures, the stiffness matrix $\boldsymbol{K}$ and its inverse $\boldsymbol{K}^{-1}$ will be symmetrical.

Let us denote by $p = |\boldsymbol{\alpha}|$ the number of the external point loads with directional uncertainty. Here we assume that all uncertain variables α_i, $i \in \{1, 2, \ldots, p\}$ are statistically independent. An uncertain load with magnitude f_i, $i \in \{1, 2, \ldots, p\}$ can be written in terms of two orthogonal loads. For simplicity we assume that one load is defined in the horizontal x direction, and the other in the vertical y direction:

$$\boldsymbol{f}_i = \left\{ f_i^x(f_i, \alpha_i) = f_i \,\boldsymbol{cos}(\alpha_i),\; f_i^y(f_i, \alpha_i) = f_i \,\boldsymbol{sin}(\alpha_i) \right\} \tag{6.31}$$

Let us assume that the two-dimensional design domain is discretised by $e^x \times e^y$ square elements. According to the usual convention, we construct the $2 \times (e^x + 1) \times (e^y + 1)$ dimensional sparse load vector $\boldsymbol{F}$ with maximum $2 \times p$ nonzero entries such that odd entries of the vector correspond to horizontal loads and even entries to vertical loads.

6.5 Optimization of Cantilever Beam with Uncertain Loads

In this chapter, we present a new benchmark problem with two directionally uncertain external loads, to show the essence of the solution searching process. The example, shown in Figure 6.5, is a cantilever beam, with a ground structure of $80\,mm \times 40\,mm \times 1\,mm$ and two unit loads acting in the bottom-middle and bottom-end positions witch are denoted by $f_i, i \in \{1, 2\}$ from left to right in the given order.

We suppose, that the uncertain load directions form an arc around the nominal loads: $240^\circ = 270^\circ - 30^0 \le \alpha_i \le 270^\circ + 30^\circ = 300^\circ$, $i \in \{1, 2\}$. The Young's modulus is $E_0 = 1$, the Poisson's ratio is $\nu = 0.3$, and the starting volume fraction, used in the nominal problem solving process, is $\varphi_0 = 0.25$. The penalization power is $p = 3$ and we use sensitivity filtering with filter radius $r_{min} = 1.5$. In this example, our goal is the following: we try to find a volume-fraction-minimal solution which is invariant to the uncertain load directions with tolerance $\tau = 1.05$.

The constant 1.05 allows maximum 5% fluctuation in the compliance space around the nominal compliance c_0. The compliance-minimal nominal solution, where $c_0 = 278.29$, is presented in Figure 6.6. In the algorithm, the nominal solution with $\{\alpha_1, \alpha_2\} = \{270^\circ, 270^\circ\}$ will be the starting base of the optimal solution searching process. In Figure 6.7, we show the shape of the compliance-minimal nominal solution. In Figure 6.8, to illustrate the change of the compliances on the set of the feasible load directions, we show the generated compliance values on the grid points of a 61×61 grid using the nominal stiffness matrix $\boldsymbol{K}_0$. This grid is fine enough to get good quality approximations for the load direction sets: $240^\circ \le \alpha_i \le 300^\circ$, $i \in \{1, 2\}$.

The worst load directions of the nominal design with $\{\alpha_1^*, \alpha_2^*\} = \{240^\circ, 249^\circ\}$ and $c^* = 320.36$ are shown in Figure 6.9. In the worst direction $\tau^* = 1.15$ which is not satisfies the requirement, therefore the nominal design is a not acceptable solution of the problem.

The starting base of the optimization with the nominal volume fraction $\varphi_0 = 0.25$ is presented in Figure 10. In Figure 11, we show the best shape of the first iteration with $\varphi = 0.300$. The worst load direction after the first iteration is shown in Figure 12 with $\{\alpha_1^*, \alpha_2^*\} = \{249^\circ, 256^\circ\}$ and $c^* = 295.75$. In the worst direction $\tau^* = 1.06$ which is not tolerable (but not so bad) value. The starting base of the second iteration with two "dangerous" load direction sets and the result of the second iteration with $\varphi = 0.301$ is presented in Figure 13-14. In Figure 15, we show the worst load directions after the second iteration $\{\alpha_1^*, \alpha_2^*\} = \{246^\circ, 254^\circ\}$ and $c^* = 292.49$. In the worst direction $\tau^* = 1.05$ which satisfies the requirement therefore the algorithm terminates. In Figure 16, we show the generated optimal compliance values on the grid points after two iterations.

When we try to compare the optimal design with the nominal design, which was presented in Figure 6.7, then our first impression is that these designs are practically the same. Fortunately, the unified visualization of the nominal and robust compliances may help to detect the real differences between the nominal and optimal shapes and understand the rearrangement mechanism of the optimal solution searching process (see Figure 16).

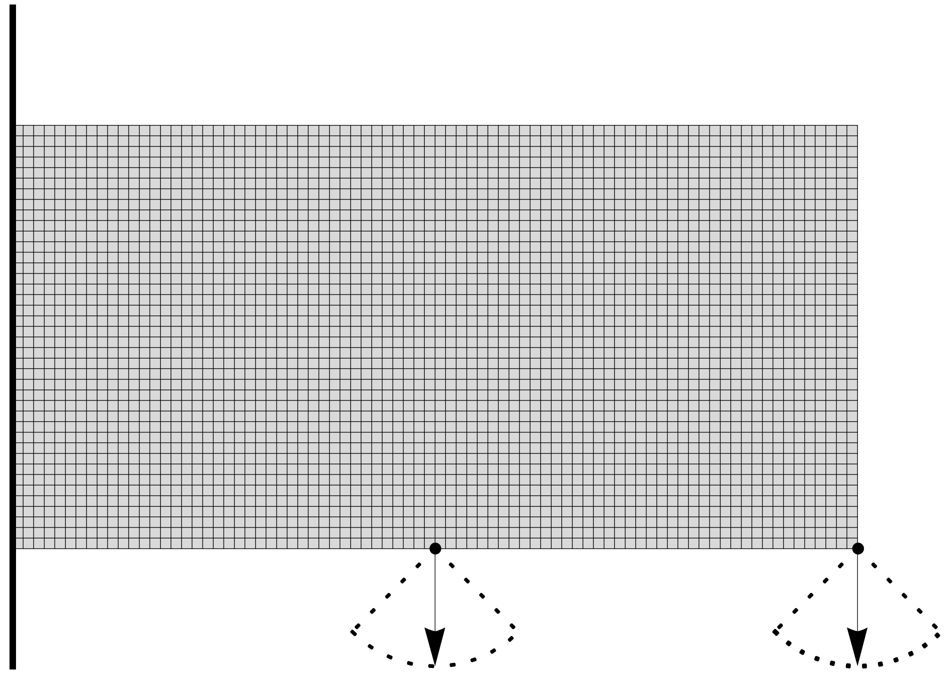

Figure 6.5: The design domain, boundary conditions, and external loads with directional uncertainty for the optimization of a cantilever beam

The result of the optimization is a more balanced shape, which tries to reach a nearly flat one according to the given tolerance.

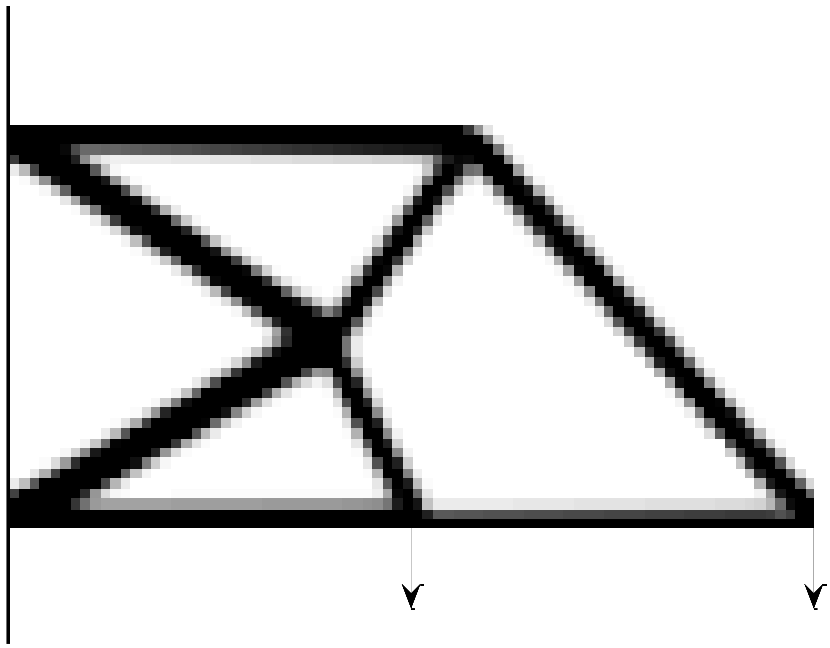

Figure 6.6: The compliance-minimal nominal solution with nominal compliance $c_0 = 278.29$

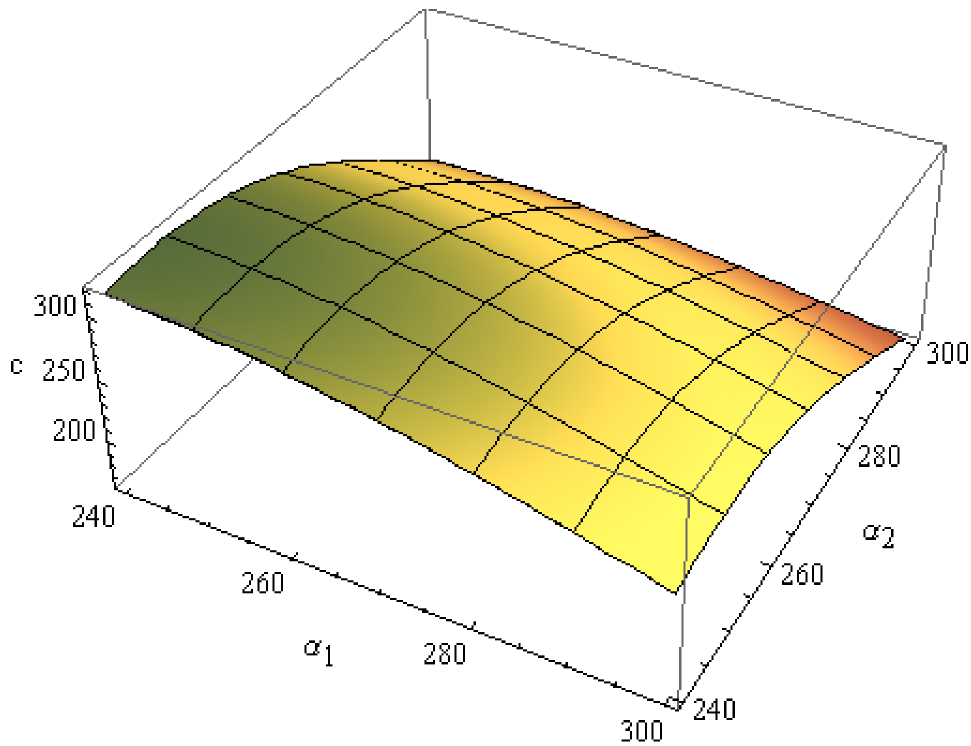

Figure 6.7: The compliance shape of the nominal solution on the set of the feasible load directions

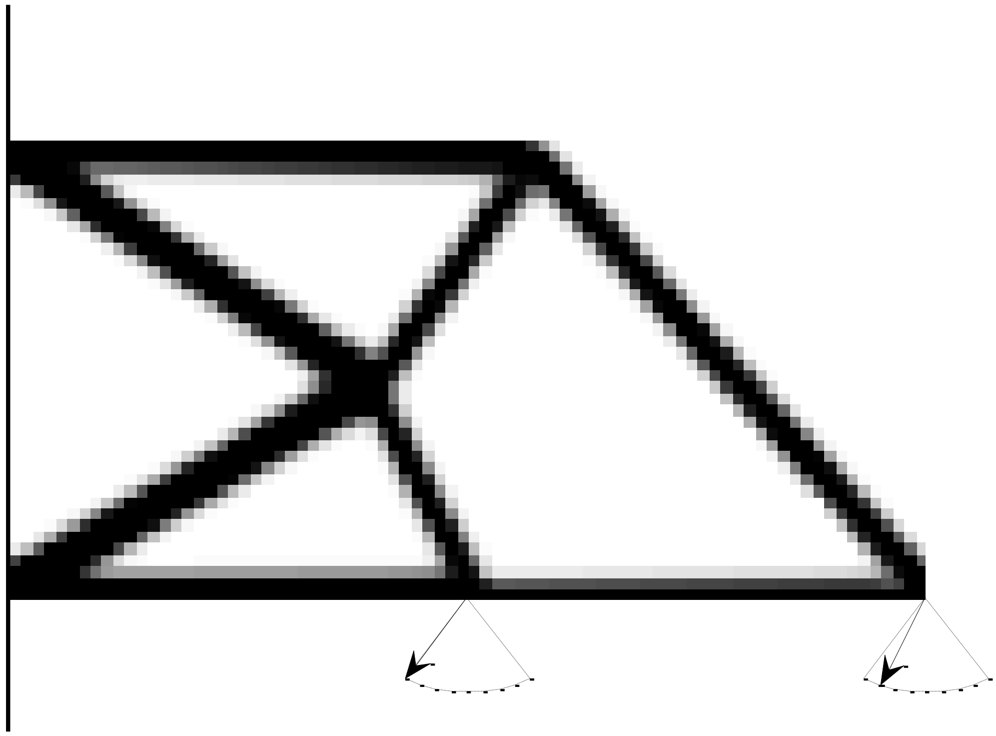

Figure 6.8: The worst load directions of the nominal design

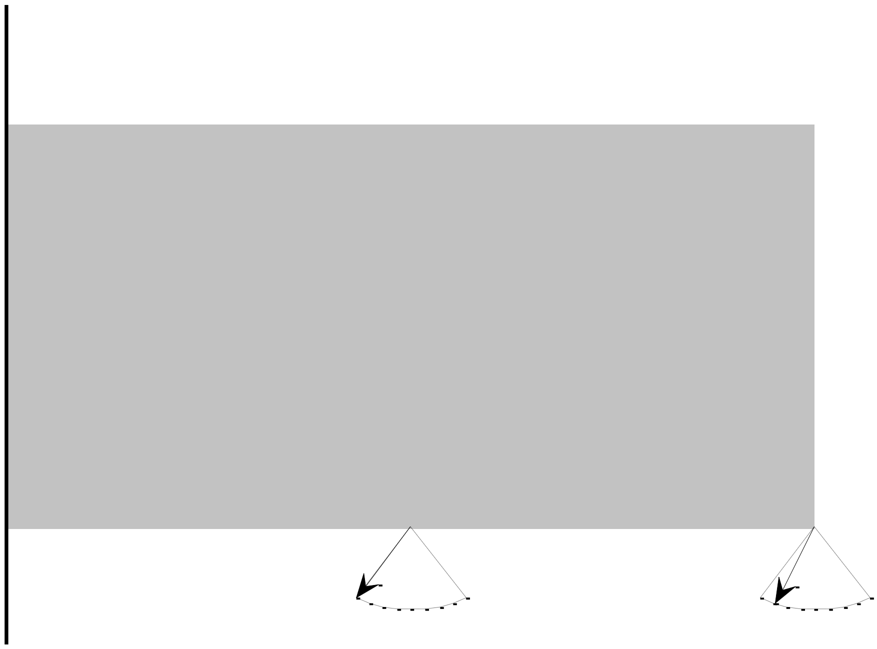

Figure 6.9: The starting base of the optimization

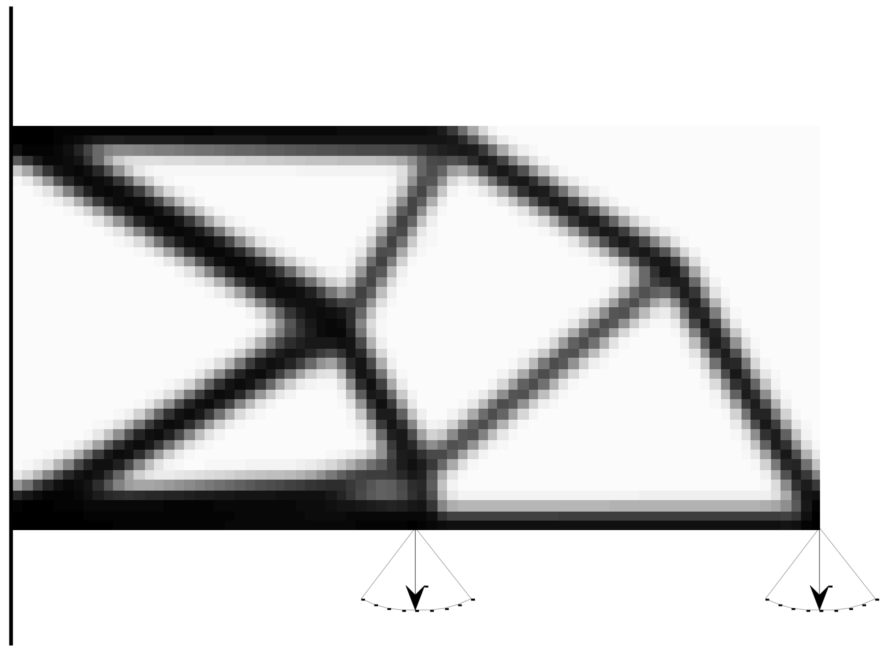

Figure 6.10: The best shape after the first iteration on the set of the feasible loads

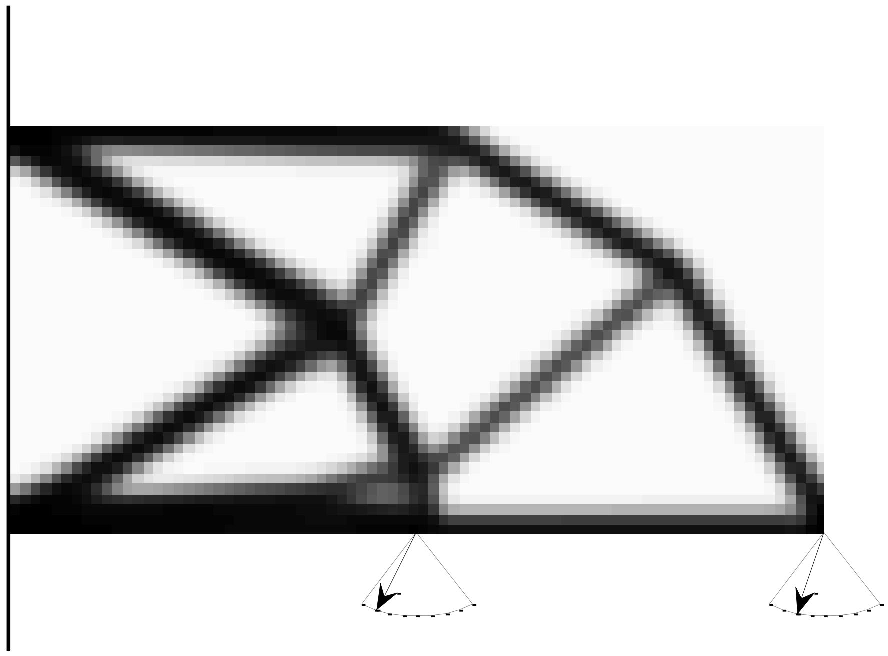

Figure 6.11: The worst direction after the first iteration

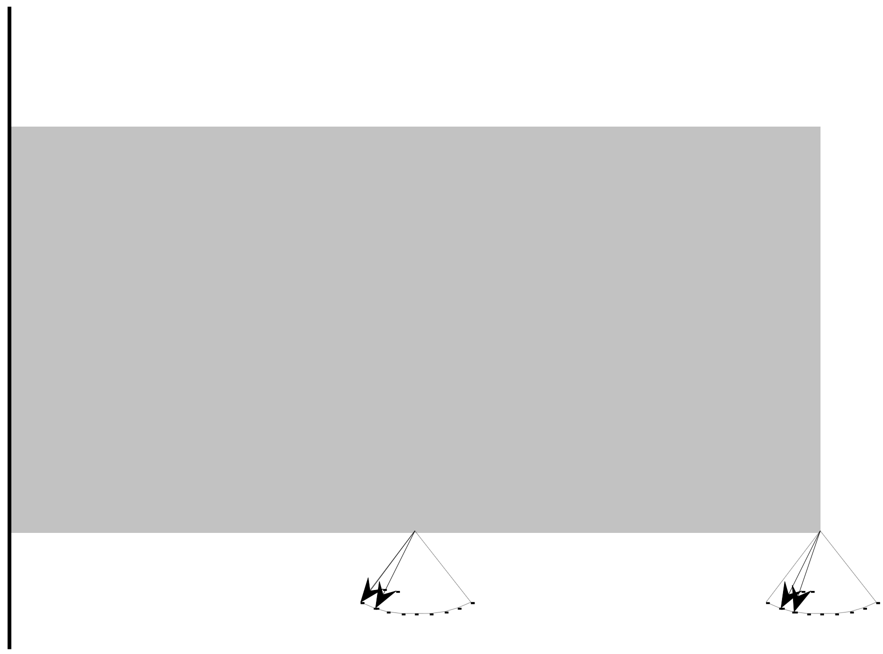

Figure 6.12: The starting base of the second iteration with two bad directions

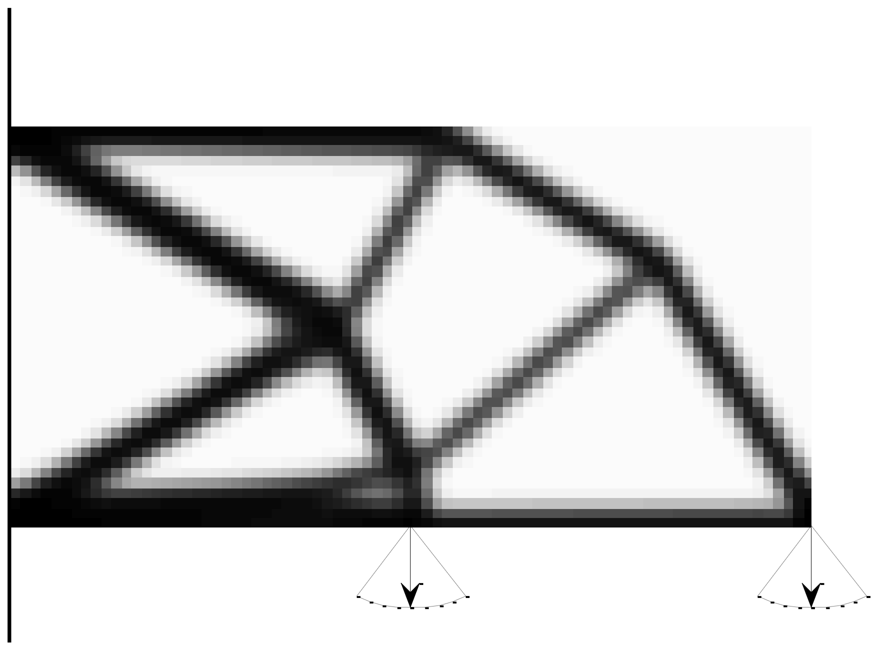

Figure 6.13: The optimal shape after the second iteration

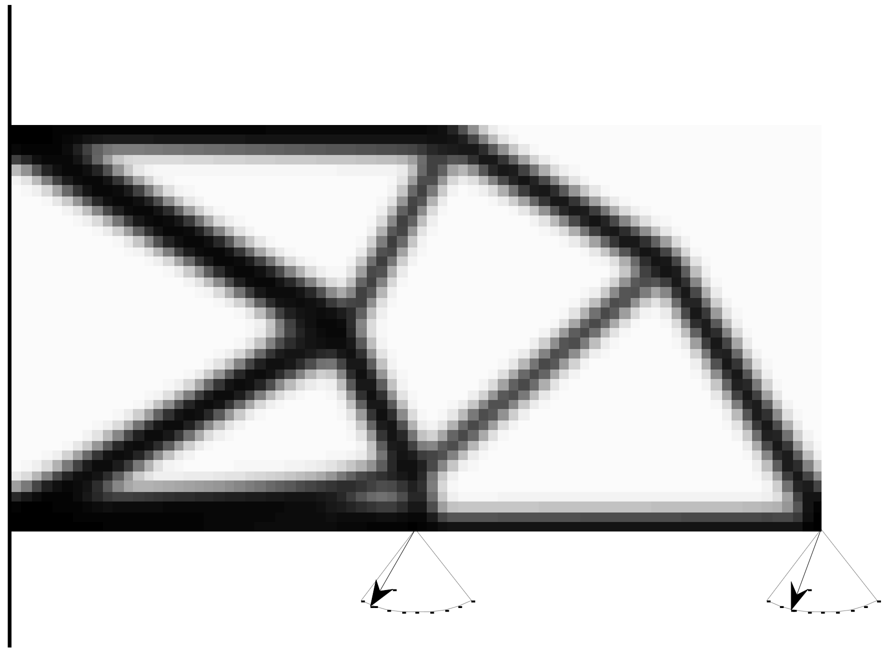

Figure 6.14: The tolerable worst load direction after the second iteration

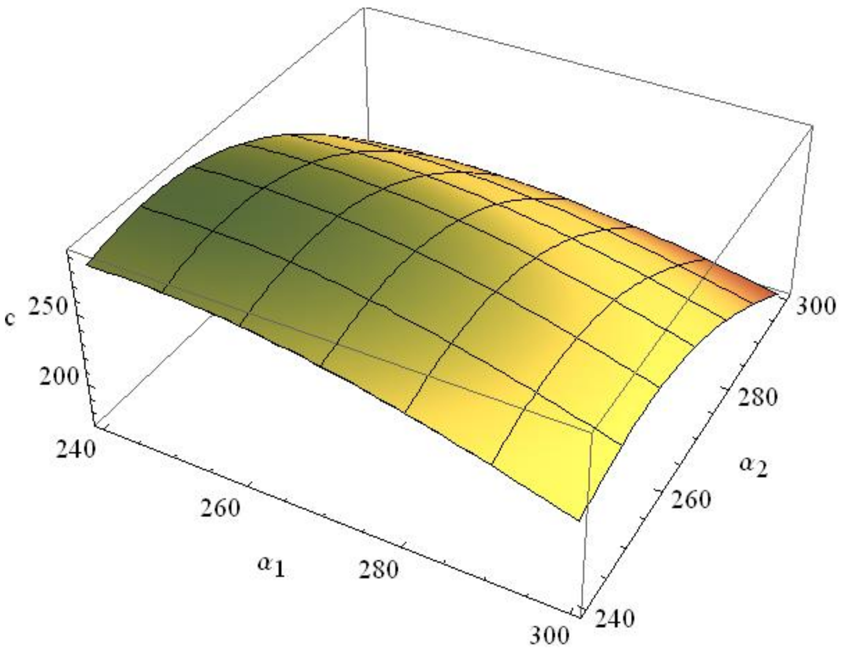

Figure 6.15: The optimal compliance shape after the second iteration

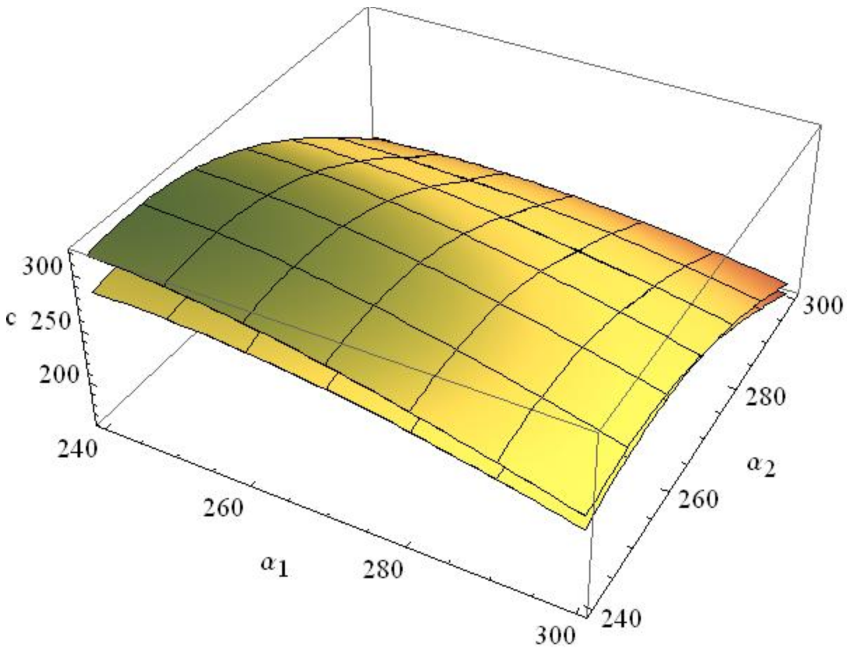

Figure 6.16: The common plot of the nominal and optimal compliance shapes

As a conclusion, we could be note, that the results good demonstrate the fact, that the optimal design searching process, using the presented simple and easy-to-understand rearrangement strategy, finds a really good solution very quickly with a very small volume fraction increment $\Delta\varphi = 0.051$.

When we solved the problem, the MATLAB package was used, which is an excellent developing tool with a state-of-the-art optimization solver and other useful features, for example, for sparse matrix manipulations and visualizations. In the presented study, a very general and stable optimization tool, namely, the **fmincon** solver was used to solve the multi-load volume fraction minimization problem and the worst-load-direction searching sub-problems. The applied structural model (SIMP), with straightforward modifications, based on the 88 lines long efficient topology optimization code developed in MATLAB by (Andreassen, 2011).

The final conclusion of this chapter will be the following statements. The essence of the novel conception is independent from the theoretical description of the directional uncertainty, which may be either *probabilistic* (stochastic) or *possibilistic* (fuzzy).

In the presented *unified* (*non-probabilistic and non-possibilistic*) approach, the varying load directions are described by quadratic constrains, which define a spherical region around the nominal loads. Naturally the applied load direction handling method can be replaced by any other uncertainty representation form, which can be described by an appropriate combination of linear or quadratic (linearizable) relations.

The result of the optimization is a design with minimum performance measure which is invariant to the investigated uncertainty type and satisfies the constraints with a given tolerance. In order to demonstrate the viability, variability and efficiency of the proposed new approach, we presented problem-specific models and algorithms with benchmark results for cross-section size optimization of 2D truss structures with displacement and stress constraints and topology optimization of 2D continuum structures with worst compliance constraint. In each case, the computational cost of extension of the standard deterministic model to a model which scopes with uncertain load directions is comparable with its deterministic equivalent.

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