Készült az ERFP – DD2002 – HU – B – 01 szerzősésszámú projekt támogatásával

C h a p t e r 2

Plane Buckling of Struts

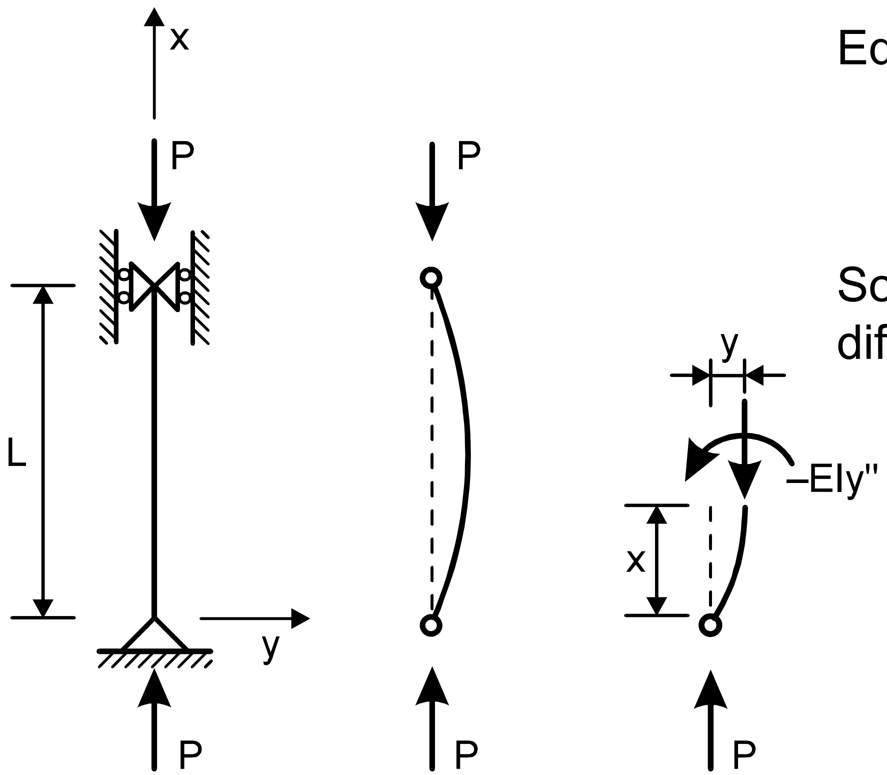
Iványi: Stability

Készült az ERFP – DD2002 – HU – B – 01 szerzősésszámú projekt támogatásával

2.1. Critical Load of the Euler-column (Equilibrium Method)

2.1.1 Fundamental Solution

Compression load:



Euler-column [1744]

Internal resisting moment: $M_x = -EI \cdot y''$

Equilibrium equation: $EI \cdot y'' + P \cdot y = 0$

$$k^2 = \frac{P}{EI} \qquad y'' + k^2 \cdot y = 0$$

Solution of the linear homogeneous differential equation:

$$y = K \cdot e^{m \cdot x} \qquad K \cdot e^{m \cdot x} \cdot (m^2 + k^2) = 0$$

$$m = \pm k \cdot \mathrm{i} \qquad y = C_1 \cdot e^{k \cdot x \cdot \mathrm{i}} + C_2 \cdot e^{-k \cdot x \cdot \mathrm{i}}$$

$$e^{\pm k \cdot x \cdot i} = \cos kx \pm \mathrm{i} \cdot \sin kx \qquad A = C_1 \cdot \mathrm{i} - C_2 \cdot \mathrm{i};$$

$$B = C_1 + C_2,$$

$$\boxed{y = A \cdot \sin kx + B \cdot \cos kx}$$

Boundary conditions:

$$x = 0 \quad \rightarrow \quad y = 0;$$
$$x = L \quad \rightarrow \quad y = 0.$$

First of these conditions:

$$B = 0 \qquad y = A \cdot \sin kx$$

Second of these conditions:

$$A \cdot \sin kL = 0$$

Trivial solution: $A = 0$

Critical solution: $\sin kL = 0$

$$k \cdot L = n \cdot \pi \qquad n = 1,2,...$$

$$P = \frac{n^2 \cdot \pi^2 \cdot EI}{L^2} \qquad y = A \cdot \sin \frac{n \cdot \pi \cdot x}{L}$$

$$n = 1: \boxed{P_E = \frac{\pi^2 \cdot EI}{L^2}} \qquad \boxed{y = A \cdot \sin \frac{\pi x}{L}}$$

Effective length: $l = L$

Tension load:

$$EI \cdot y'' - P \cdot y = 0; \quad k^2 = \frac{P}{EI}$$

$$y = K \cdot e^{m \cdot x}$$

$$K \cdot e^{m \cdot x} \cdot (m^2 - k^2) = 0$$

$$m = \pm k$$

$$y = C_1 \cdot e^{k \cdot x} + C_2 \cdot e^{-k \cdot x}$$

$$e^{\pm k \cdot x} = \mathrm{ch}\, kx \pm \mathrm{sh}\, kx$$

$$A = C_1 - C_2;$$
$$B = C_1 + C_2,$$

$$y = A \cdot \mathrm{sh}\, kx + B \cdot \mathrm{ch}\, kx$$

No stability problem!

2.1.2 Effect of Approximations

(a) Axial deformations:

Direct axial compression:

$$\varepsilon_0 = \frac{P_{kr} \cdot L}{EA}$$

$$M(x) = -EI \cdot \frac{1+\varepsilon_0}{\rho} = -EI \cdot (1+\varepsilon_0) \cdot y'' = P \cdot y$$

$$y'' + k^2 \cdot y = 0$$

$$k^2 = \frac{P}{EI \cdot (1+\varepsilon_0)}$$

Boundary conditions:

$$x = 0 \quad \rightarrow \quad y = 0;$$

$$x = L \cdot (1+\varepsilon_0) \quad \rightarrow \quad y = 0.$$

Condition of buckling:

$$\sin(1+\varepsilon_0)kL = 0$$

Eigenvalue:

$$(1+\varepsilon_0) \cdot k \cdot L = n \cdot \pi$$

$$n = 0,1,2,...$$

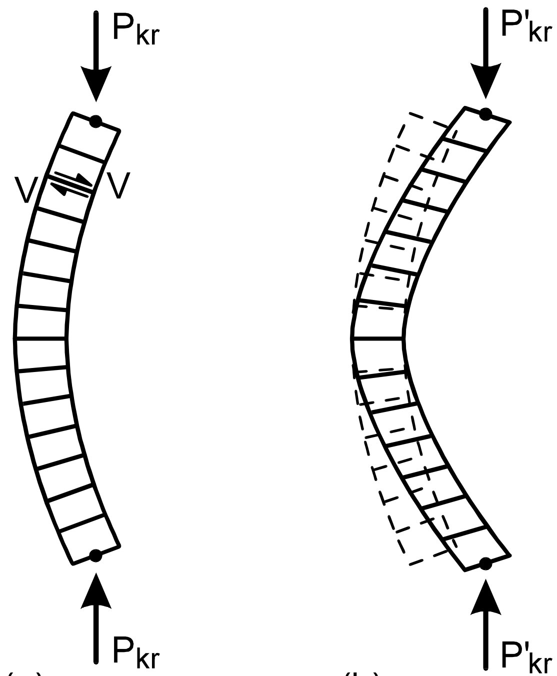
$$n = 1:$$

$$P_{kr} = \pi^2 \cdot \frac{EI}{L^2} \cdot \frac{1}{1 - \frac{P_{kr}}{EA}}$$

$$\frac{P_{kr}}{EA} = \frac{\sigma_{kr}}{E}$$

“52” steel: 0.17%

(b) Effect of Shear Deformations on Critical Loads



(a) (b)

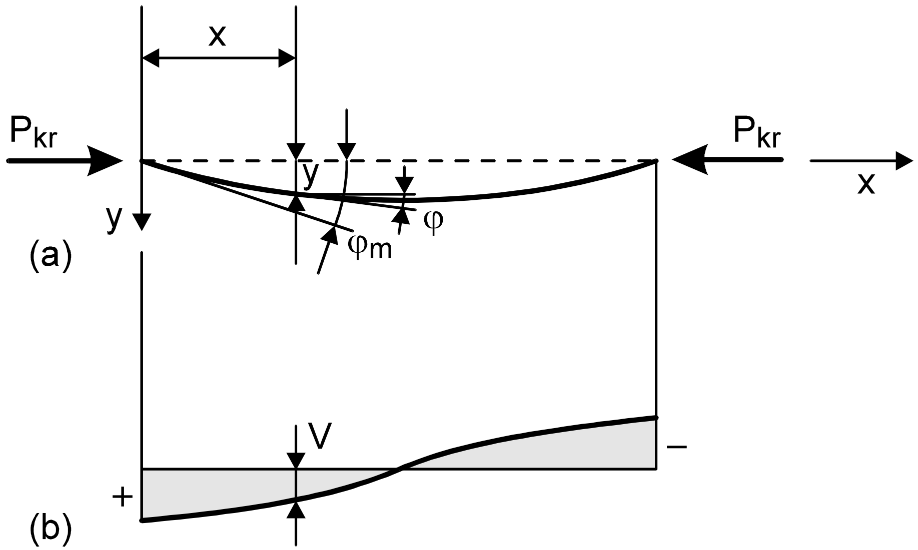
The applied load P_{kr} will have components transverse to the bent longitudinal axis, thus introducing into the member shear forces V as shown. It will in turn produce additional deformation due to shear.

$$V = P_{kr} \cdot \sin\varphi \qquad V = P_{kr} \cdot tan\varphi = P_{kr} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = P_{kr} \cdot y'$$

$$V_{\max} = P_{kr} \cdot tan\varphi_m$$

Curvature: $$\frac{1}{R_t} = \frac{\bar{\rho}}{GA} \cdot \frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{\bar{\rho}}{GA} \cdot \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \cdot P_{kr}$$

[Timoshenko, Gere, 1961]



Cross-section	$\bar{\rho}$
circle	32/27
rectangle	6/5
I, shear parallel to web	A_t / A_w
I, shear parallel to flange	$1{,}2A_t / A_{fl}$

Approximation:

$$\frac{1}{R} = \frac{d^2 y}{dx^2} = -\frac{M}{EI} - \frac{\bar{\rho}}{GA} \cdot \frac{d^2 y}{dx^2} \cdot P_{kr}$$

$$M = P_{kr} \cdot y$$

$$EI \cdot \left(1 + \frac{\bar{\rho}}{GA} \cdot P_{kr}\right) \cdot y'' + P_{kr} \cdot y = 0$$

$$\left(1 + \frac{\bar{\rho}}{GA}\right) \cdot P_{kr}$$

$$P_{kr} = \frac{\pi^2 \cdot EI}{L^2 \cdot \left(1 + \frac{\bar{\rho}}{GA}\right) \cdot P_{kr}} = \frac{\pi^2 \cdot EI}{(\gamma \cdot L)^2}$$

$$\gamma^2 = 1 + \bar{\rho} \cdot \frac{P_{kr}}{GA} \qquad \boxed{\frac{\bar{\rho} \cdot P_{kr}}{GA}}$$

For solid cross-sections this effect is unimportant, amounting to a reduction of a fraction of 1%. For lattice truss cross-sections this effect is of significant importance.

(c) Effect of Large Deformations

Differential Equation: $$y'' + \frac{P}{EI} \cdot y = 0$$

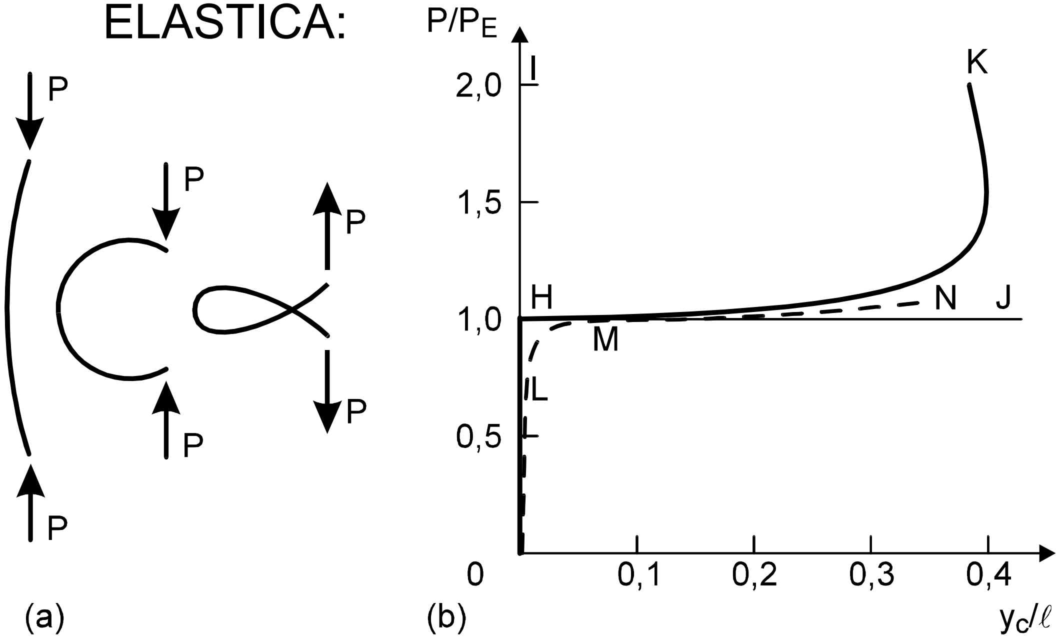
Correct expression for curvature:

$$M_x = -\frac{EI \cdot y''}{(1 + y'^2)^{3/2}}$$

Hence the DE:

$$y'' + \frac{P}{EI} \cdot (1 + y'^2)^{3/2} \cdot y = 0$$

ELASTICA:



Load – deformation curve

2.2. Higher-Order Differential Equation for Columns

$$EI \cdot y'' + P \cdot y = -V \cdot x + M_A$$

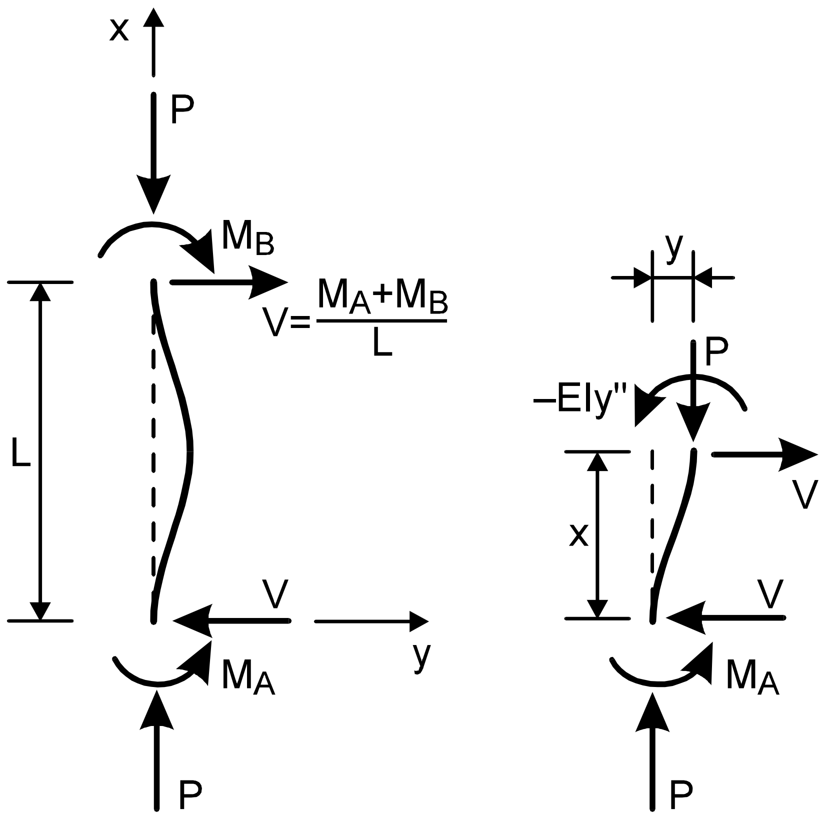
$$V = \frac{M_A + M_B}{L}$$

$$EI \cdot y^{IV} + P \cdot y'' = 0$$

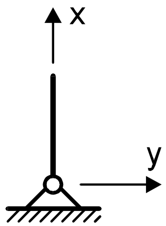
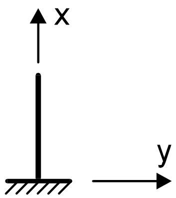
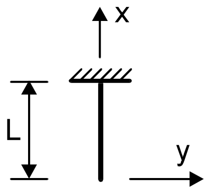
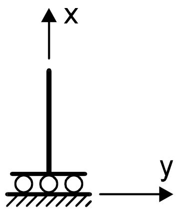
$$k^2 = P / EI$$

$$y^{IV} + k^2 \cdot y'' = 0$$

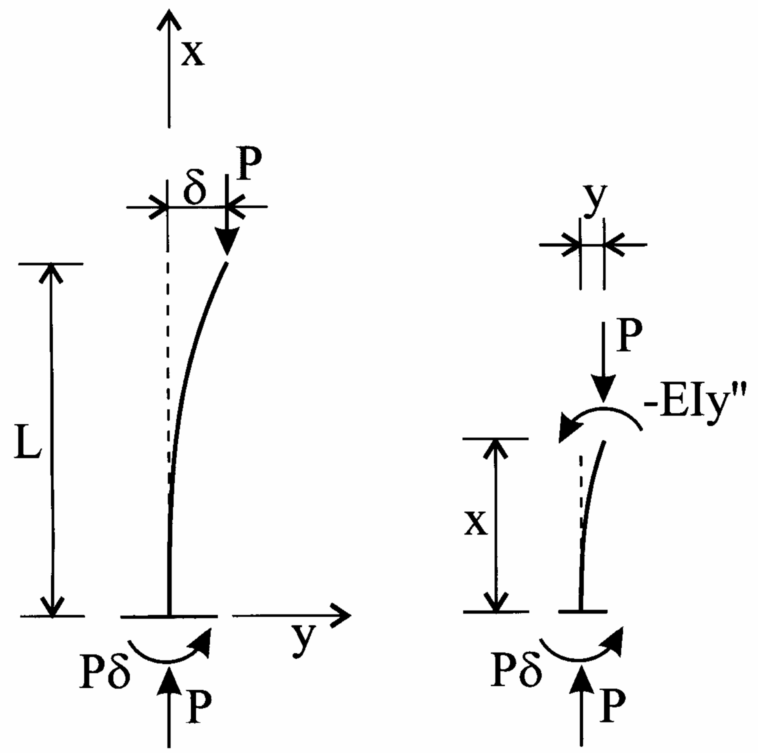
$$y = C_1 \cdot \sin kx + C_2 \cdot \cos kx + C_3 \cdot x + C_4$$



Different boundary conditions for DE:

hinged		$x = 0$	$y = 0$ $y'' = 0$
fixed		$x = 0$	$y = 0$ $y' = 0$
free		$x = 0$	$y'' = 0$ $y''' + k^2 \cdot y' = 0$
movable-fixed		$x = 0$	$y' = 0$ $y''' = 0$

Fixed – Free Column



Boundary conditions:

$$x = 0 \quad \rightarrow \quad y = 0; \quad y' = 0;$$
$$x = L \quad \rightarrow \quad y'' = 0; \quad y''' + k^2 \cdot y' = 0.$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin kL & \cos kL & 0 & 0 \\ 0 & 0 & k^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \mathbf{0}$$

Non-trivial solution: $\cos kL = 0$

$$k \cdot L = \frac{(2n-1)\cdot\pi}{2} \qquad n = 1,2,3...$$

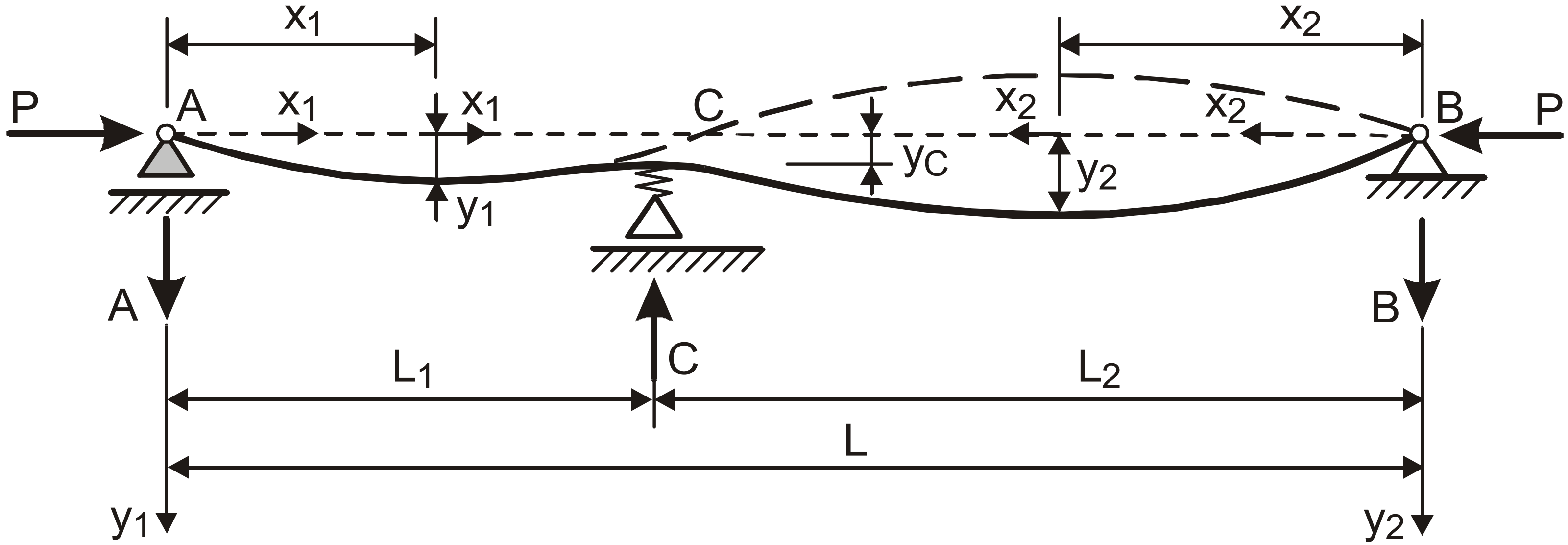
$$\boxed{P_{kr} = \frac{\pi^2 \cdot EI}{4L^2}}$$

Effective length: $l = 2L$

2.3. Effective Length of Compression Members

2.3.1 Intermediate Restraints

(a) Single Restraint



Equilibrium condition:

$$A + B + C = 0$$

$$\sum M_C = A \cdot L_1 - B \cdot L_2 = 0$$

$$A = -C \cdot \frac{L_2}{L} \qquad B = -C \cdot \frac{L_1}{L}$$

Left-hand side Equation:

$$-C \cdot \frac{L_2}{L} \cdot x_1 + P \cdot y_1 + EI \cdot y_1'' = 0$$

Right-hand side Equation:

$$-C \cdot \frac{L_1}{L} \cdot x_2 + P \cdot y_2 + EI \cdot y_2'' = 0$$

$$k^2 = P/EI$$

$$y_1 = A_1 \cdot \sin kx_1 + B_1 \cdot \cos kx_1 + \frac{C}{P} \cdot \frac{L_2}{L} \cdot x_1;$$

$$y_2 = A_2 \cdot \sin kx_2 + B_2 \cdot \cos kx_2 + \frac{C}{P} \cdot \frac{L_1}{L} \cdot x_2.$$

Boundary conditions:

$$x_1 = 0 \quad \rightarrow \quad y_1 = 0$$

$$x_2 = 0 \quad \rightarrow \quad y_2 = 0$$

$$B_1 = B_2 = 0$$

$$x_1 = L_1 \quad \rightarrow \quad y_1 = C/c$$

$$x_2 = L_2 \quad \rightarrow \quad y_2 = C/c$$

$$y_1' = -y_2'$$

$$\frac{C}{P} \cdot \left(\frac{L_1 \cdot L_2}{L} - \frac{P}{c} \right) + A_1 \cdot \sin kL_1 = 0;$$

$$\frac{C}{P} \cdot \left(\frac{L_1 \cdot L_2}{L} - \frac{P}{c} \right) + A_2 \cdot \sin kL_2 = 0;$$

$$\frac{C}{P} + A_1 \cdot k \cdot \cos kL_1 + A_2 \cdot k \cdot \cos kL_2 = 0,$$

$$\det \begin{bmatrix} \frac{L_1 \cdot L_2}{L} - \frac{P}{c} & \sin kL_1 & 0 \\ \frac{L_1 \cdot L_2}{L} - \frac{P}{c} & 0 & \sin kL_2 \\ 1 & k \cdot \cos kL_1 & k \cdot \cos kL_2 \end{bmatrix} = 0.$$

If $L_1 = L_2 = L/2$

$$\sin^2 k\frac{L}{2} - \left(\frac{L}{4} - \frac{P}{c} \right) \cdot k \cdot \sin kL = 0$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\sin k\frac{L}{2} \cdot \left[\sin k\frac{L}{2} - 2k \cdot \left(\frac{L}{4} - \frac{P}{c} \right) \cdot \cos k\frac{L}{2} \right] = 0$$

First solution: $\sin k\frac{L}{2} = 0$

$$k \cdot L/2 = n \cdot \pi$$

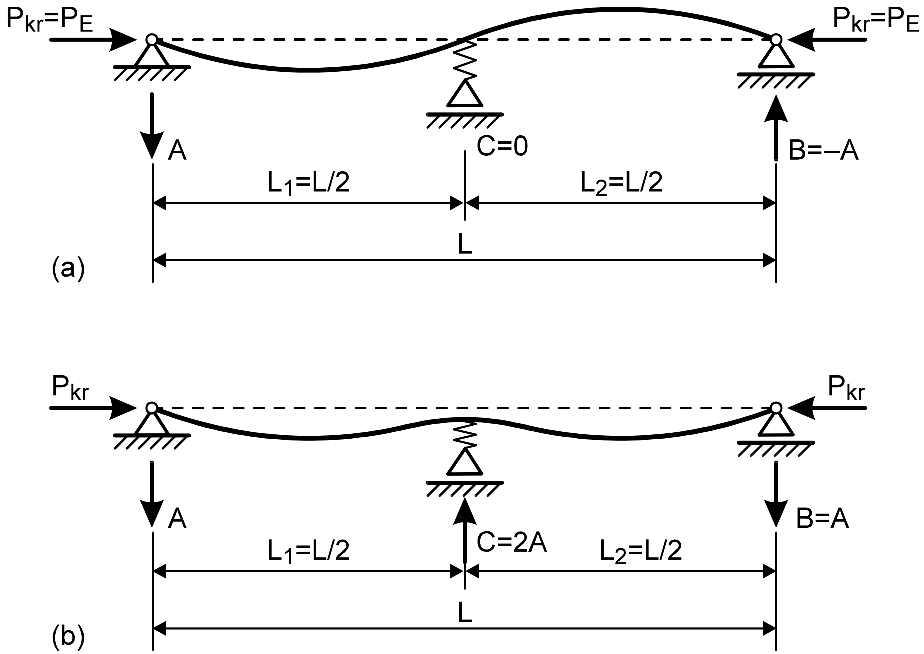
$$\boxed{P_{kr} = \frac{n^2 \cdot \pi^2 \cdot EI}{(L/2)^2}}$$

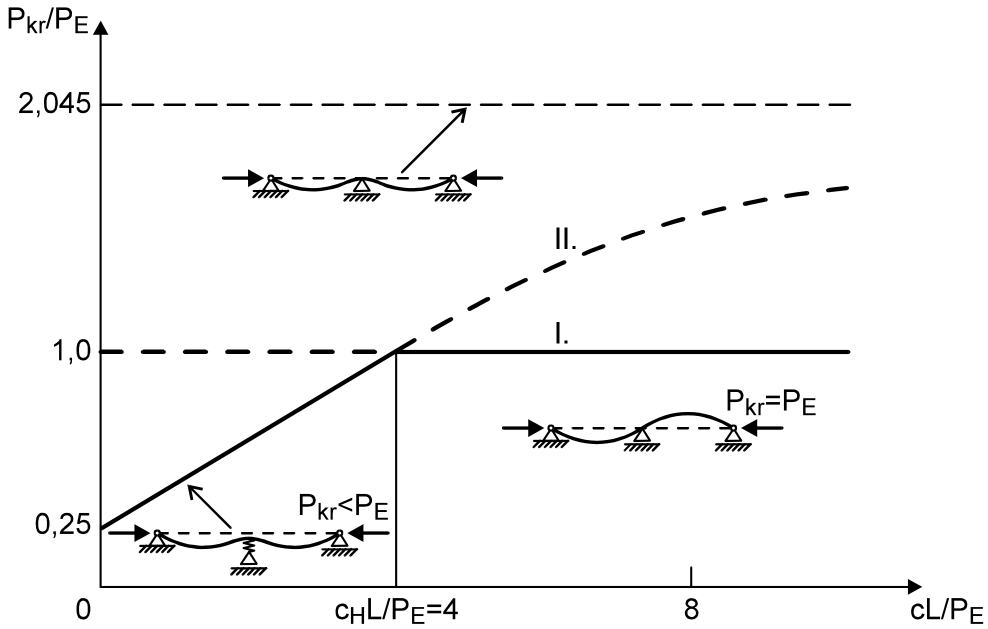
$$\frac{L}{4} - \frac{P}{c} \neq 0 \qquad C = 0 \qquad A_1 = -A_2$$

Second solution:

$$\operatorname{tg} k\frac{L}{2} = \frac{k \cdot L}{2} \cdot \left(1 - \frac{4P}{c \cdot L}\right) \longrightarrow A_1 = A_2 \quad \begin{cases} c = 0 \\ c \to \infty \longrightarrow \operatorname{tg} k\frac{L}{2} = \frac{k \cdot L}{2} \end{cases}$$

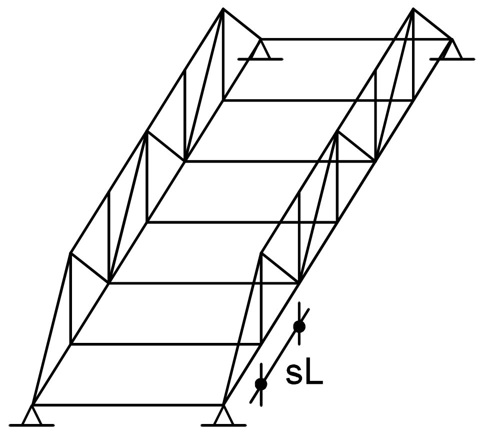
$$\boxed{P_{kr} = 2{,}045 P_E}$$

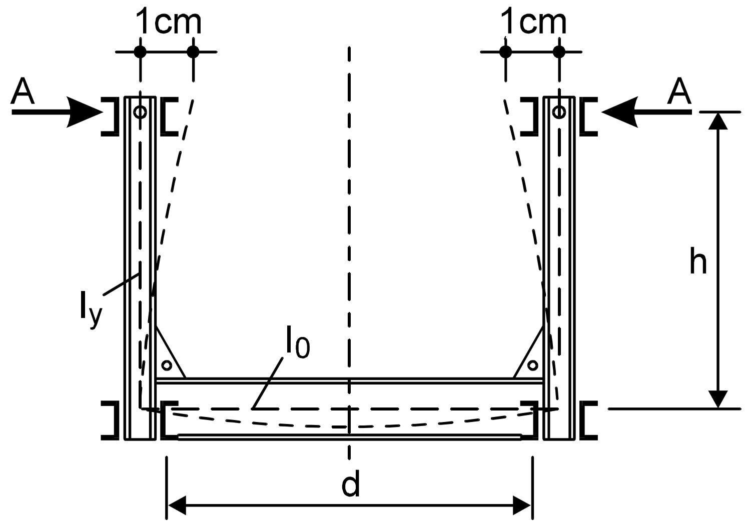




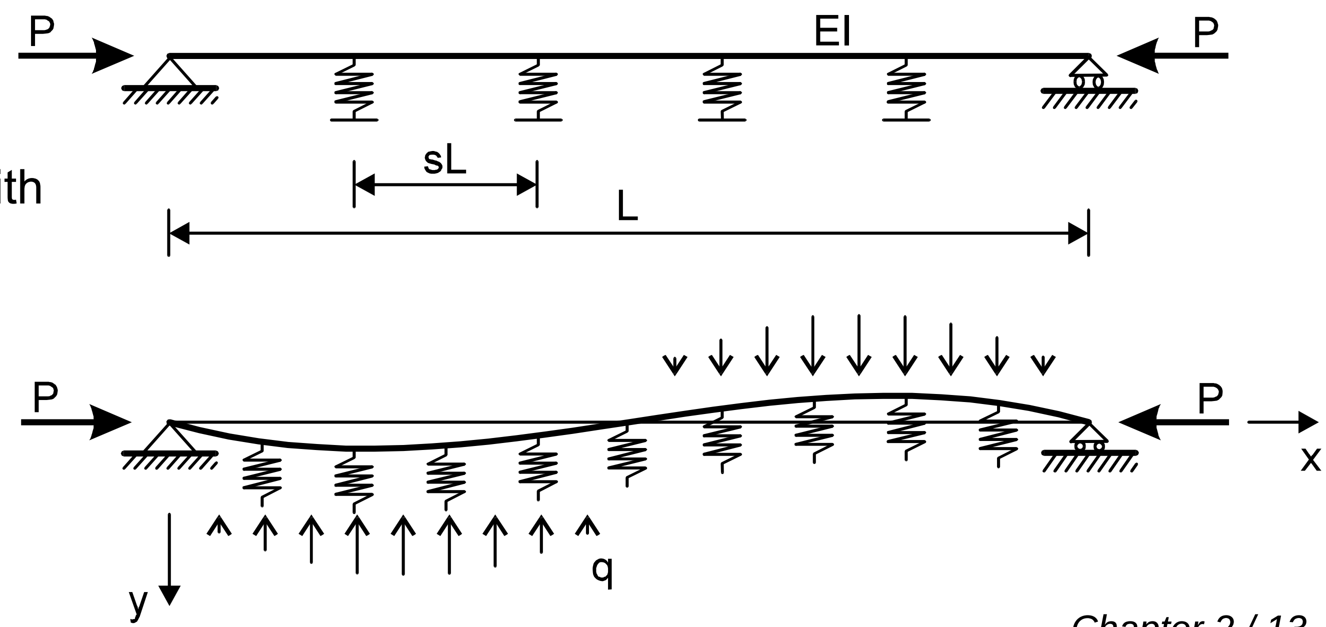
(b) Continuous Restraint

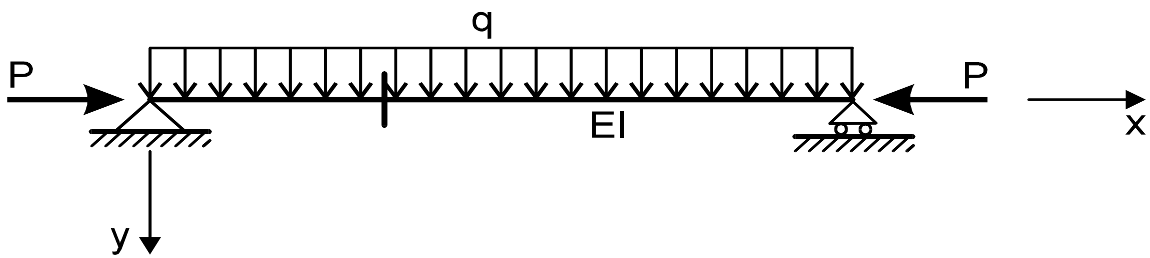
Half-through Bridge



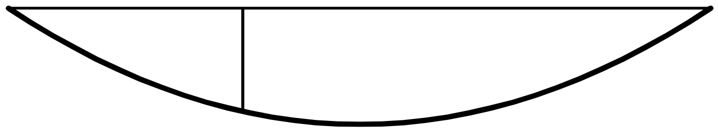


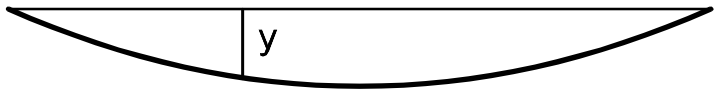
Compression Member with Continuous Restraint [Engesser, 1884]





Compression Member with Transversal Load


Bending Moment from Transversal Loadl



Deflection

$$c = \frac{A}{s \cdot L} \qquad q = c \cdot y$$

$$y = y_0 \cdot \sin \frac{m\pi x}{L}$$

Equilibrium condition:

$$EI \cdot y_0 \cdot \left(\frac{m \cdot \pi}{L}\right)^4 \cdot \sin\frac{m\pi x}{L} - P \cdot y_0 \cdot \left(\frac{m \cdot \pi}{L}\right)^2 \cdot \sin\frac{m\pi x}{L} + c \cdot y_0 \cdot \sin\frac{m\pi x}{L} = 0$$

$$M_x = M_0 + P \cdot y$$

$$EI \cdot \left(\frac{m \cdot \pi}{L}\right)^4 - P \cdot \left(\frac{m \cdot \pi}{L}\right)^2 + c = 0$$

$$EI \cdot y'' + P \cdot y + M_0 = 0$$

$$EI \cdot y^{IV} + P \cdot y'' + q = 0$$

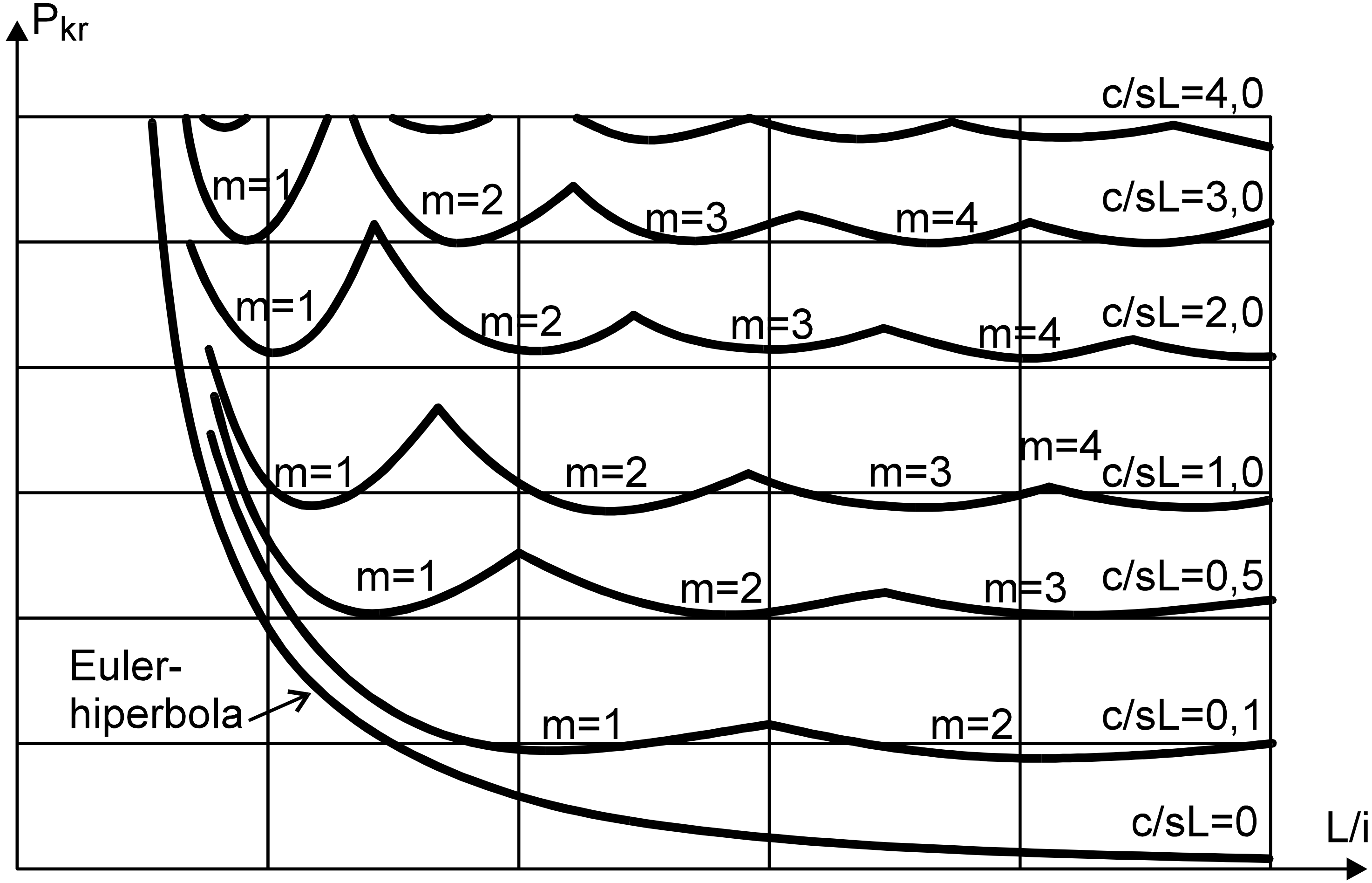
$$P_{kr} = \left(\frac{m \cdot \pi}{L}\right)^2 \cdot EI + c \cdot \left(\frac{L}{m \cdot \pi}\right)^2 = \frac{\pi^2 \cdot EI}{L^2} \cdot \left(m^2 + \frac{L^4 \cdot c}{m^2 \cdot \pi^4 \cdot EI}\right)$$

$$EI \cdot y^{IV} + P \cdot y'' + c \cdot y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}m}\left(m^2+\frac{L^4\cdot c}{m^2\cdot\pi^4\cdot EI}\right)=2m-\frac{2L^4\cdot c}{m^3\cdot\pi^4\cdot EI}=0$$

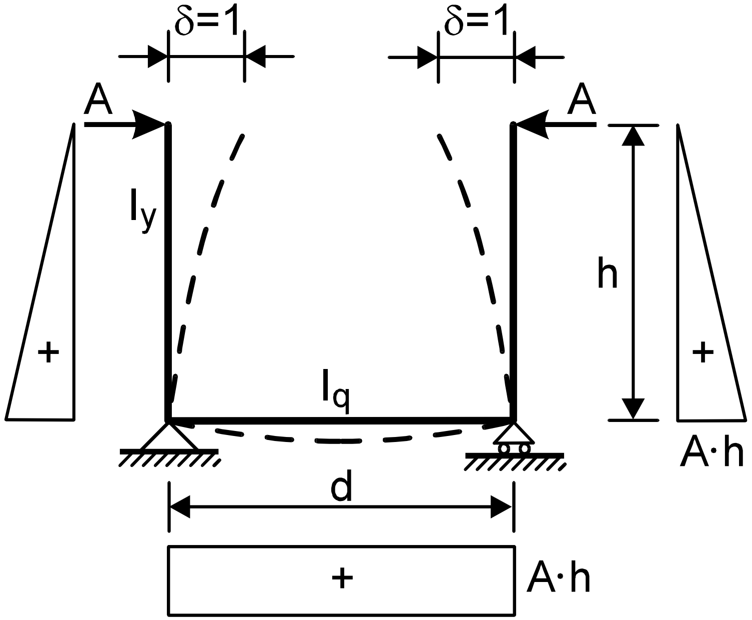
$$m^2=\frac{L^2}{\pi^2}\cdot\sqrt{\frac{c}{EI}}$$

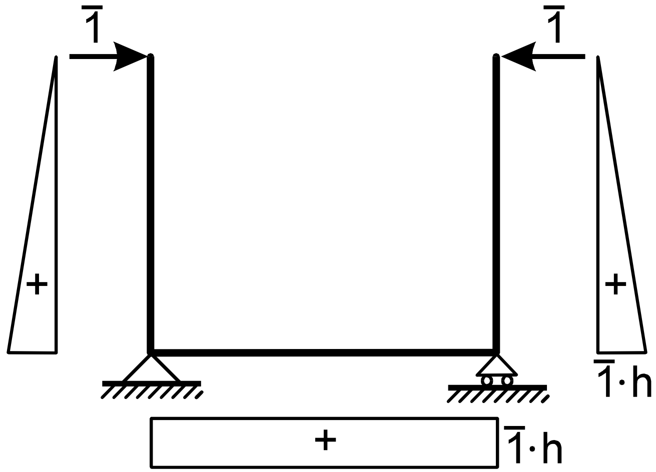
$$\boxed{P_{kr}=2\cdot\sqrt{c\cdot EI}}$$



[Chwalla, 1927]

Half-through Bridge (U-frame)





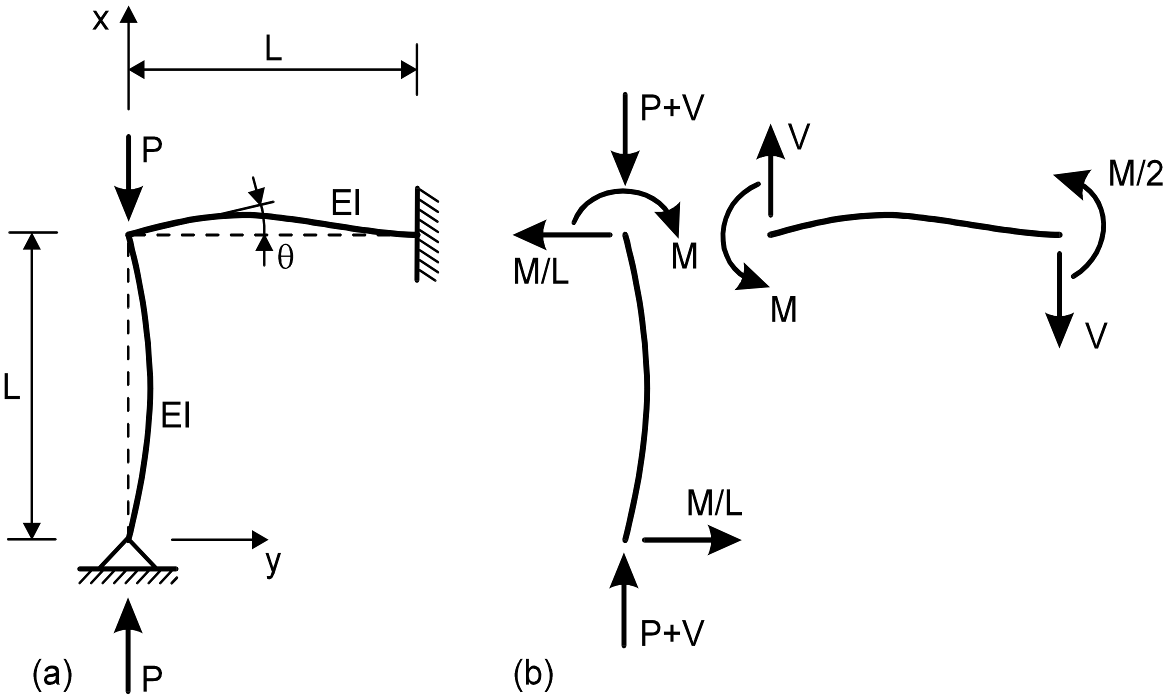
δ – horizontal deflection:

$$2\delta = 2 = \frac{A \cdot h^2 \cdot d}{EI_0} + 2 \cdot \frac{1}{3} \cdot \frac{A \cdot h^3}{EI_y}$$

U-frame stiffness:

$$A = \frac{1}{\dfrac{d \cdot h^2}{2EI_0} + \dfrac{h^3}{3EI_y}}$$

2.3.2 Elastically Restrained Column



Equilibrium condition:

$$-EI \cdot y'' - P \cdot y + \frac{M \cdot x}{L} = 0$$

$$k^2 = P / EI$$

$$y'' + k^2 \cdot y = \frac{M}{EI} \cdot \frac{x}{L}$$

$$y = A \cdot \sin kx + B \cdot \cos kx + \frac{M}{P} \cdot \frac{x}{L}$$

Boundary conditions:

$$x = 0 \quad \rightarrow \quad y = 0 \qquad B = 0$$

$$x = L \quad \rightarrow \quad y = 0 \qquad A = -\frac{M}{P} \cdot \frac{1}{\sin kL}$$

$$y = \frac{M}{P} \cdot \left(\frac{x}{L} - \frac{\sin kx}{\sin kL} \right)$$

$$x = L:$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{M}{P} \cdot \left(\frac{1}{L} - \frac{k \cdot \cos kx}{\sin kL} \right)$$

$$\left. \frac{\mathrm{d}y}{\mathrm{d}x} \right|_{x=L} = \frac{M}{k \cdot EI} \cdot \left(\frac{1}{k \cdot L} - \frac{1}{\mathrm{tg}\, kL} \right)$$

$$\theta = \frac{M \cdot L}{4EI}$$

$$\frac{M \cdot L}{4EI} = -\frac{M}{k \cdot EI} \cdot \left(\frac{1}{k \cdot L} - \frac{1}{\mathrm{tg} kL} \right)$$

$$\frac{k \cdot L}{4} = -\left(\frac{1}{k \cdot L} - \frac{1}{\mathrm{tg} kL} \right)$$

$$\mathrm{tg}\, kL = \frac{4k \cdot L}{(k \cdot L)^2 + 4} \qquad k \cdot L = 3.83$$

$$P_{kr} = \frac{14.7 \cdot EI}{L^2} = \frac{\pi^2 \cdot EI}{(0.82 \cdot L)^2}$$

Effective length: $l = 0.82L$

2.4. Effect of Loading System

2.4.1 Initially Bent Columns

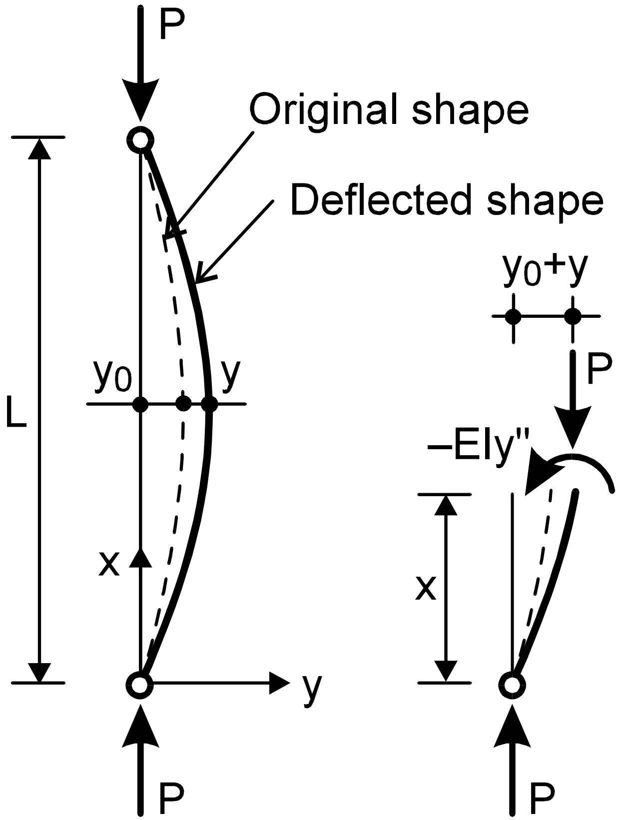
Initial deformation:

$$y_0 = e_0 \cdot \sin \frac{\pi x}{L}$$

$$EI \cdot y'' + P \cdot (y_0 + y) = 0$$

$$k^2 = P / EI$$

$$y'' + k^2 \cdot y = -k^2 \cdot e_0 \cdot \sin \frac{\pi x}{L}$$



Homogenous and particular solutions:

$$y = y_h + y_p$$

$$y_h = A \cdot \sin kx + B \cdot \cos kx$$

$$y_p = C \cdot \sin \frac{\pi x}{L} + D \cdot \cos \frac{\pi x}{L}$$

$$\left[C \cdot \left(k^2 - \frac{\pi^2}{L^2} \right) + k^2 \cdot e_0 \right] \cdot \sin \frac{\pi x}{L} + D \cdot \left(k^2 - \frac{\pi^2}{L^2} \right) \cdot \cos \frac{\pi x}{L} = 0$$

$C = \dfrac{e_0}{\dfrac{\pi^2}{k^2 \cdot L^2} - 1}$ and $D = 0$ or $k^2 = \pi^2 / L^2$

$$k^2 = \pi^2 / L^2 \longrightarrow P = \frac{\pi^2 \cdot EI}{L^2}$$

$$D = 0 \longrightarrow \frac{1}{\psi} = \alpha = \frac{P}{P_E}$$

$$P_E = \frac{\pi^2 \cdot EI}{L^2}$$

$$C = \frac{e_0}{\psi - 1} = \frac{e_0 \cdot \alpha}{1 - \alpha} \qquad y_p = \frac{e_0 \cdot \alpha}{1 - \alpha} \cdot \sin \frac{\pi x}{L}$$

$$y = A \cdot \sin kx + B \cdot \cos kx + \frac{\alpha}{1 - \alpha} \cdot e_0 \cdot \sin \frac{\pi x}{L}$$

Boundary conditions:

$$x = 0 \rightarrow y = 0 \qquad B = 0$$

$$x = L \rightarrow y = 0 \qquad A \cdot \sin kL = 0$$

$$\sin kL = 0 \qquad P = P_E \qquad A = 0$$

Bending deflection: $y = \dfrac{\alpha}{1 - \alpha} \cdot e_0 \cdot \sin \dfrac{\pi x}{L}$

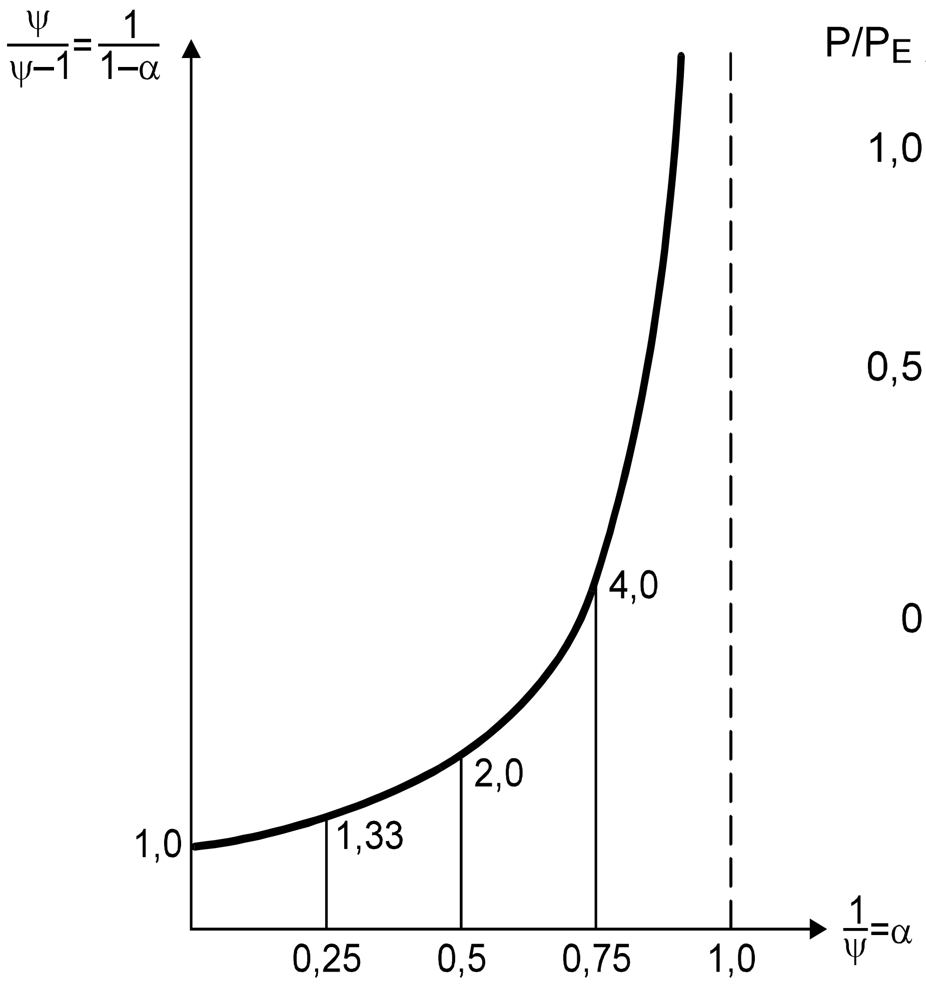
Total deflection:

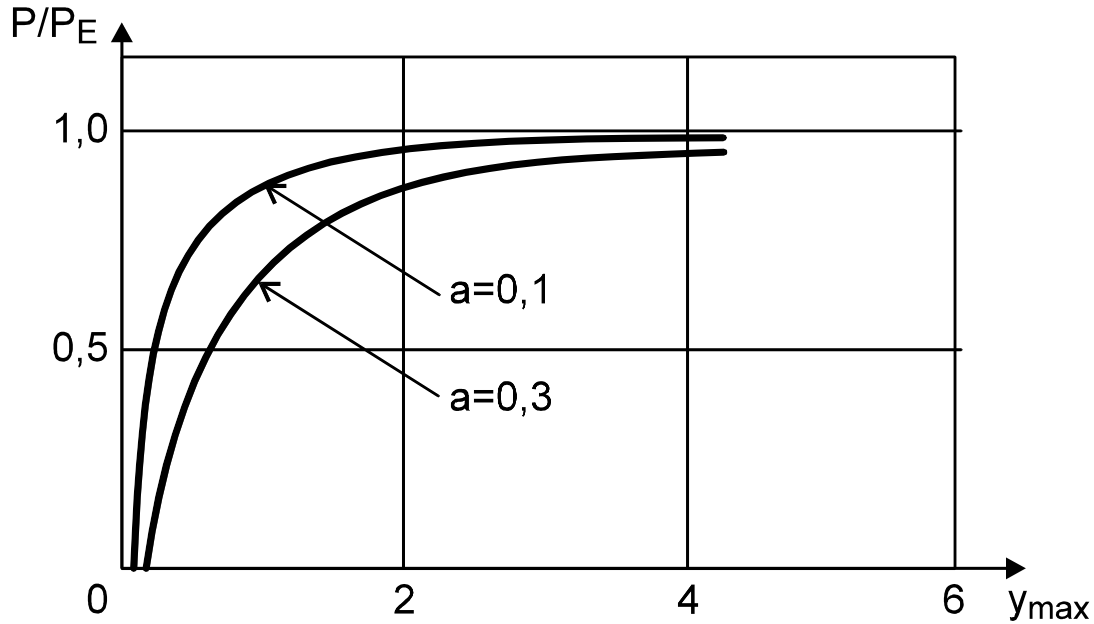
$$y_T = y_0 + y = \left(1 + \frac{\alpha}{1 - \alpha} \right) \cdot e_0 \cdot \sin \frac{\pi x}{L} = \frac{e_0}{1 - \alpha} \cdot \sin \frac{\pi x}{L}$$

Total deflection at mid-height: $e = \dfrac{e_0}{1 - \alpha} = \dfrac{e_0}{1 - \dfrac{P}{P_E}} = \dfrac{\psi}{\psi - 1} \cdot e_0$

Bending Moment:

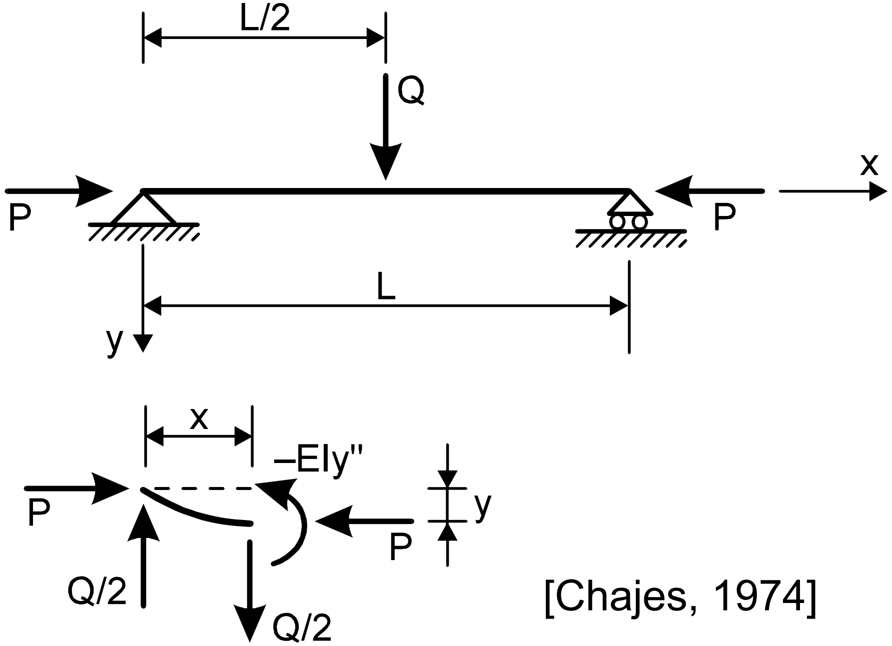
$$M_{\max} = P \cdot e = \frac{\psi}{\psi - 1} \cdot P \cdot e_0 = \frac{1}{1-\alpha} \cdot P \cdot e_0 = \frac{1}{1-\alpha} \cdot M_0$$





Load–deflection curves of initially bent columns

2.4.2 Beam-Column with Lateral Loads



[Chajes, 1974]

External Moment: $M = \frac{Q \cdot x}{2} + P \cdot y$

$$EI \cdot \frac{\mathrm{d}^2 y}{\mathrm{d} x^2} + P \cdot y = -\frac{Q \cdot x}{2} \qquad k^2 = P / EI$$

$$y = A \cdot \sin kx + B \cdot \cos kx - \frac{Q \cdot x}{2P}$$

Boundary conditions:

$$x = 0 \quad \rightarrow \quad y = 0 \qquad B = 0$$

$$x = L/2 \quad \rightarrow \quad y' = 0 \qquad A = \frac{Q}{2k \cdot P} \cdot \frac{1}{\cos \frac{kL}{2}}$$

$$y = \frac{Q}{2P \cdot k} \cdot \left[\frac{\sin kx}{\cos \frac{kL}{2}} - k \cdot x \right]$$

Midspan Deflection:

$$y_{\max} = \frac{Q}{2P \cdot k} \cdot \left[\frac{\sin \frac{kL}{2}}{\cos \frac{kL}{2}} - \frac{k \cdot L}{2} \right]$$

$$y_{\max} = \frac{Q}{2P \cdot k} \cdot (\operatorname{tg} u - u) \qquad u = k \cdot L / 2$$

$$y_{\max} = \frac{Q \cdot L^3}{48EI} \cdot \frac{3 \cdot (\operatorname{tg} u - u)}{u^3}$$

Deflection with Lateral Load:

$$y_{\max,0} = \frac{Q \cdot L^3}{48EI}$$

$$y_{\max} = y_{\max,0} \cdot \frac{3(\operatorname{tg} u - u)}{u^3}$$

$$\operatorname{tg} u = u + \frac{1}{3} \cdot u^3 + \frac{2}{15} \cdot u^5 + \frac{17}{315} \cdot u^7 + \ldots$$

$$y_{\max} = y_{\max,0} \cdot \left(1 + \frac{2}{5} \cdot u^2 + \frac{17}{105} \cdot u^4 + \ldots\right)$$

$k^2 = P / EI$ and $u = k \cdot L / 2$

$$u^2 = \frac{P \cdot L^2}{4EI} = 2,46 \frac{P}{P_E}$$

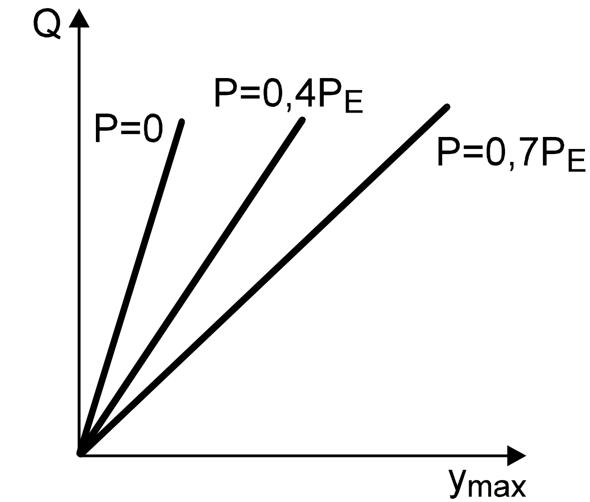
$$y_{\max} = y_{\max,0} \cdot \left[1 + 0,987 \cdot \frac{P}{P_E} + 0,998 \cdot \left(\frac{P}{P_E}\right)^2 + \ldots\right] \approx$$

$$\approx y_{\max,0} \cdot \left[1 + \frac{P}{P_E} + \left(\frac{P}{P_E}\right)^2 + \ldots\right]$$

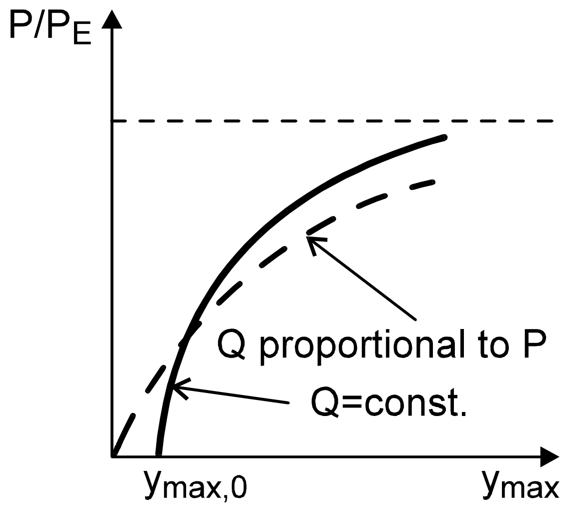
Since the sum of the geometric series inside the brackets is $1/[1 - P / P_E]$:

$$y_{\max} = y_{\max,0} \cdot \frac{1}{1 - \frac{P}{P_E}}$$

Készült az ERFP – DD2002 – HU – B – 01 szerzősésszámú projekt támogatásával



(a)



(b)

Beam-Column Load–Deflection Characteristics

Restricted Superposition:

Bending moment at midspan:

$$M_{\max} = M_0 + P \cdot \delta = \frac{Q \cdot L}{4} + P \cdot \delta$$

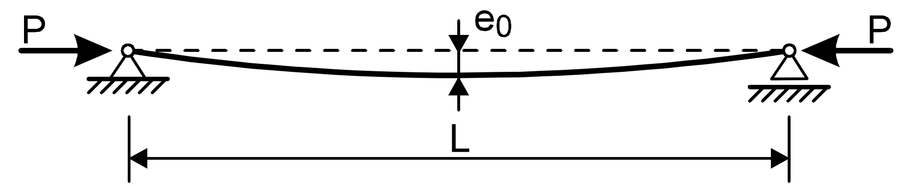
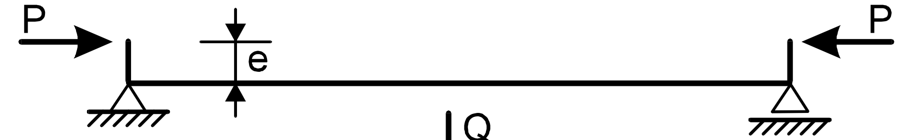
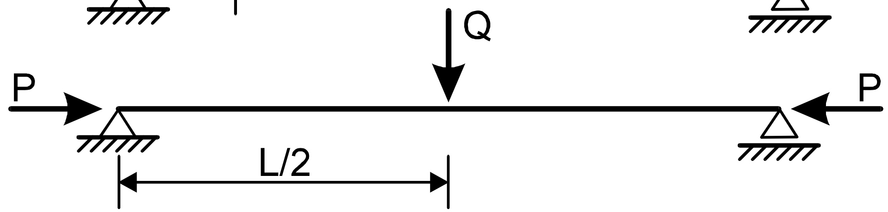
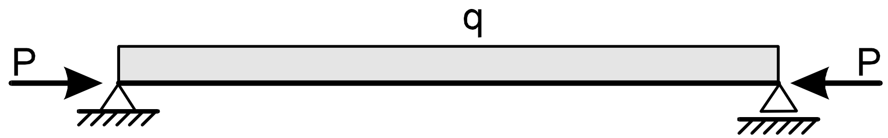
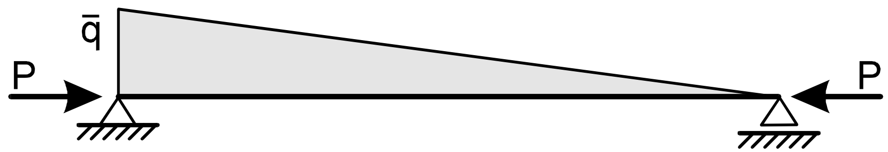
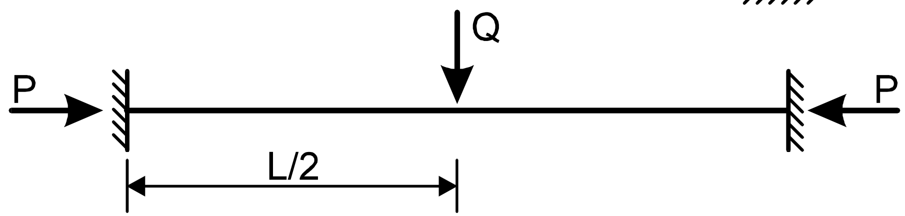
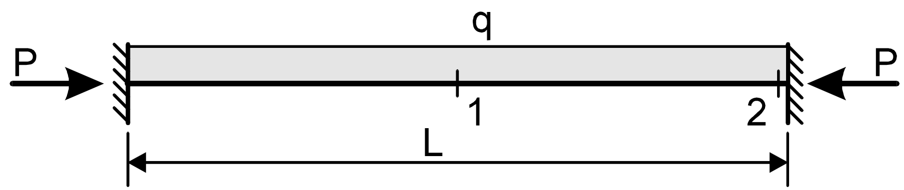
$$M_{\max} = \frac{Q \cdot L}{4} + \frac{P \cdot Q \cdot L^3}{48EI} \cdot \frac{1}{1 - \frac{P}{P_E}}$$

$$M_{\max} = M_0 \cdot \frac{1 - 0.18 \cdot \frac{P}{P_E}}{1 - \frac{P}{P_E}}$$

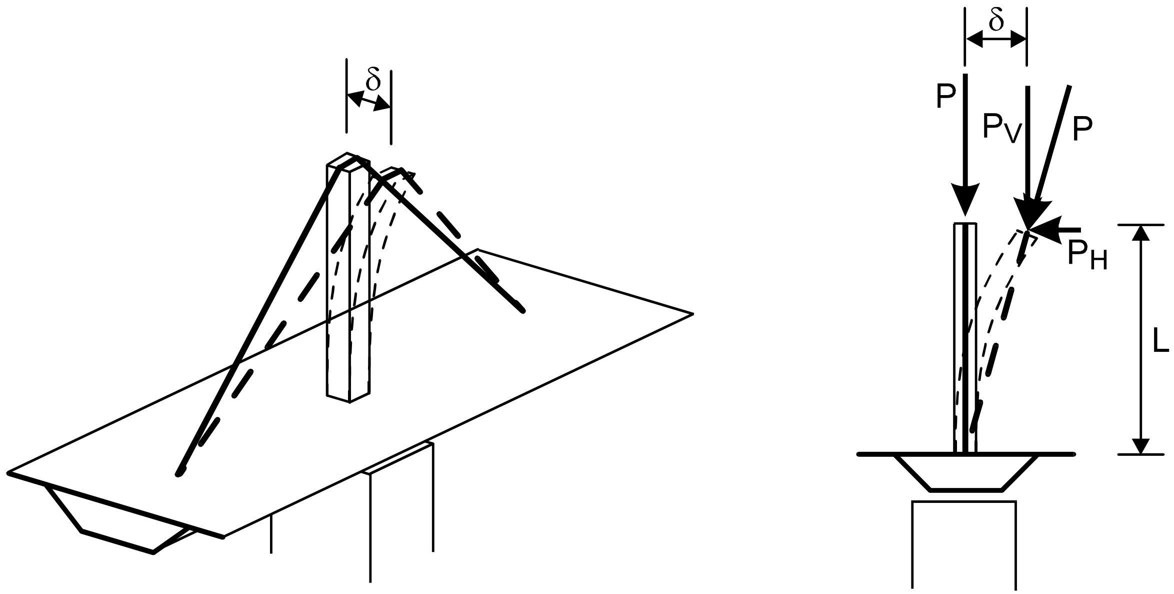
$$M_{\max} = M_0 \cdot \frac{1 - \delta \cdot \frac{P}{P_E}}{1 - \frac{P}{P_E}}$$

[Dischinger, 1937]

Készült az ERFP – DD2002 – HU – B – 01 szerzősésszámú projekt támogatásával

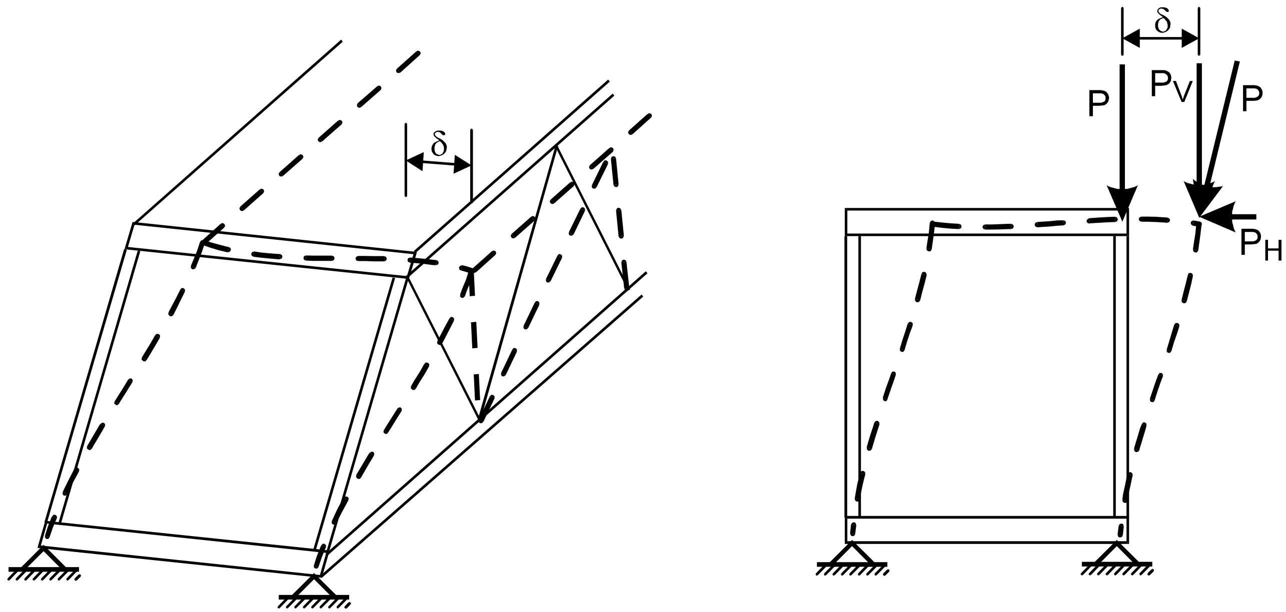
Load Cases	M_0	δ
	$P \cdot e_0$	0
	$P \cdot e$	0.273
	$\frac{Q \cdot L}{4}$	–0.189
	$\frac{q \cdot L^2}{8}$	0.0324
	$\frac{0.128 q \cdot L^2}{2}$	0.0324
	$\frac{Q \cdot L}{8}$	–0.189
	$\frac{q \cdot L^2}{24}$	“1”: +0.121 “2”: –0.362

2.4.3 Guided Load System



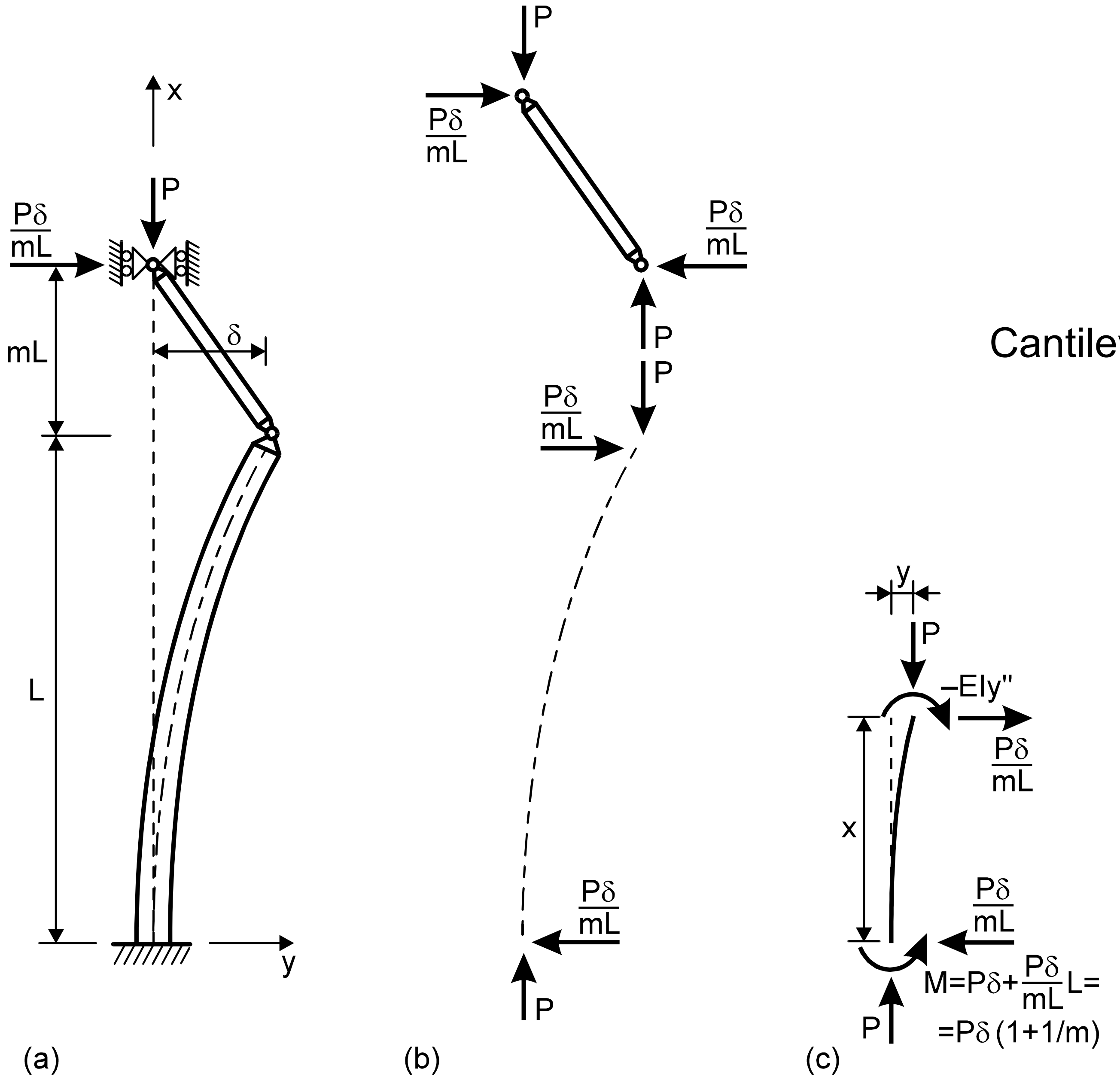
$$P_H = -P \cdot \frac{\delta}{L}$$

Pylon of Cable Stayed Bridge



Through Truss Bridge

Cantilever with Pendelum



Bending Moment:

$$M_x = P \cdot y + \frac{P \cdot \delta}{m \cdot L} \cdot x - P \cdot \delta \cdot \left(1 + \frac{1}{m}\right)$$

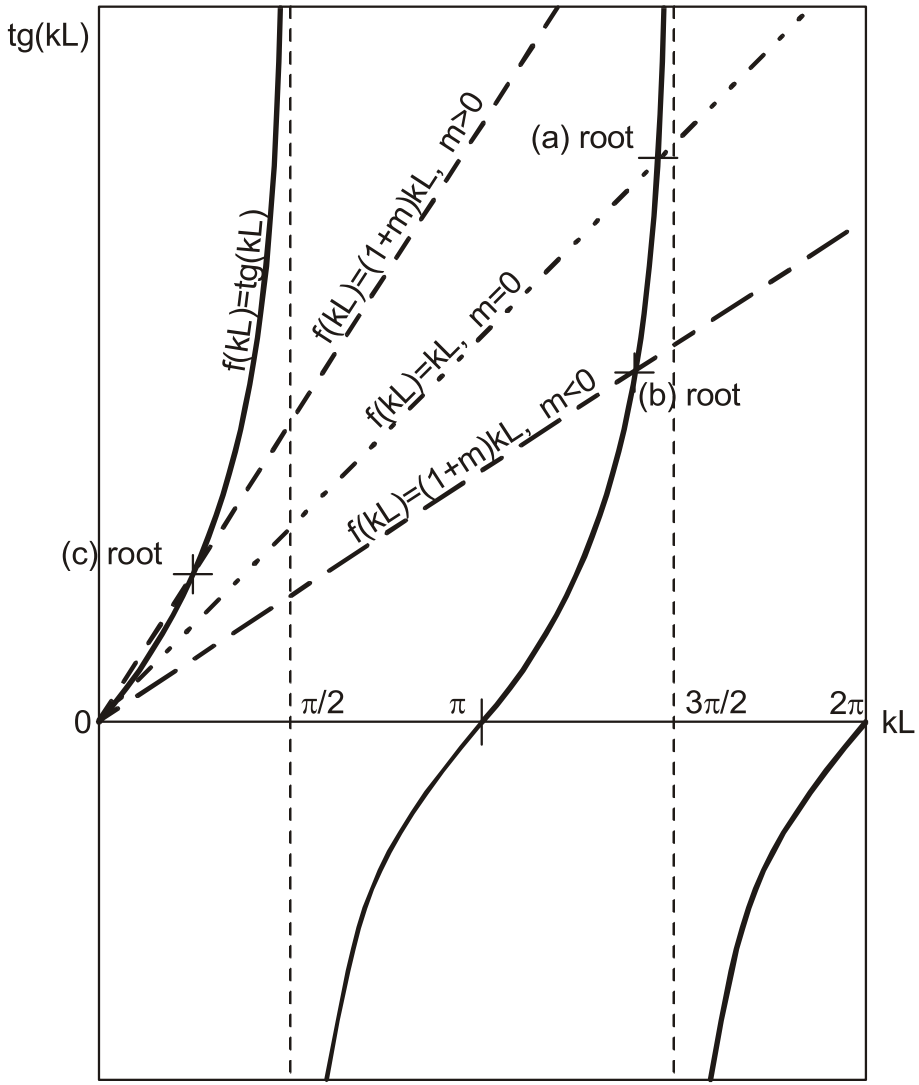
$$EI \cdot \frac{d^2 y}{dx^2} = -M_x = -P \cdot y - \frac{P \cdot \delta}{m \cdot L} \cdot x + P \cdot \delta \cdot \left(1 + \frac{1}{m}\right)$$

$$k^2 = P / EI$$

$$y'' + k^2 \cdot y = -k^2 \cdot \frac{\delta}{m \cdot L} \cdot x + k^2 \cdot \delta \cdot \left(1 + \frac{1}{m}\right)$$

$$y = A \cdot \sin kx + B \cdot \cos kx - \frac{\delta}{m \cdot L} \cdot x + \delta \cdot \left(1 + \frac{1}{m}\right)$$

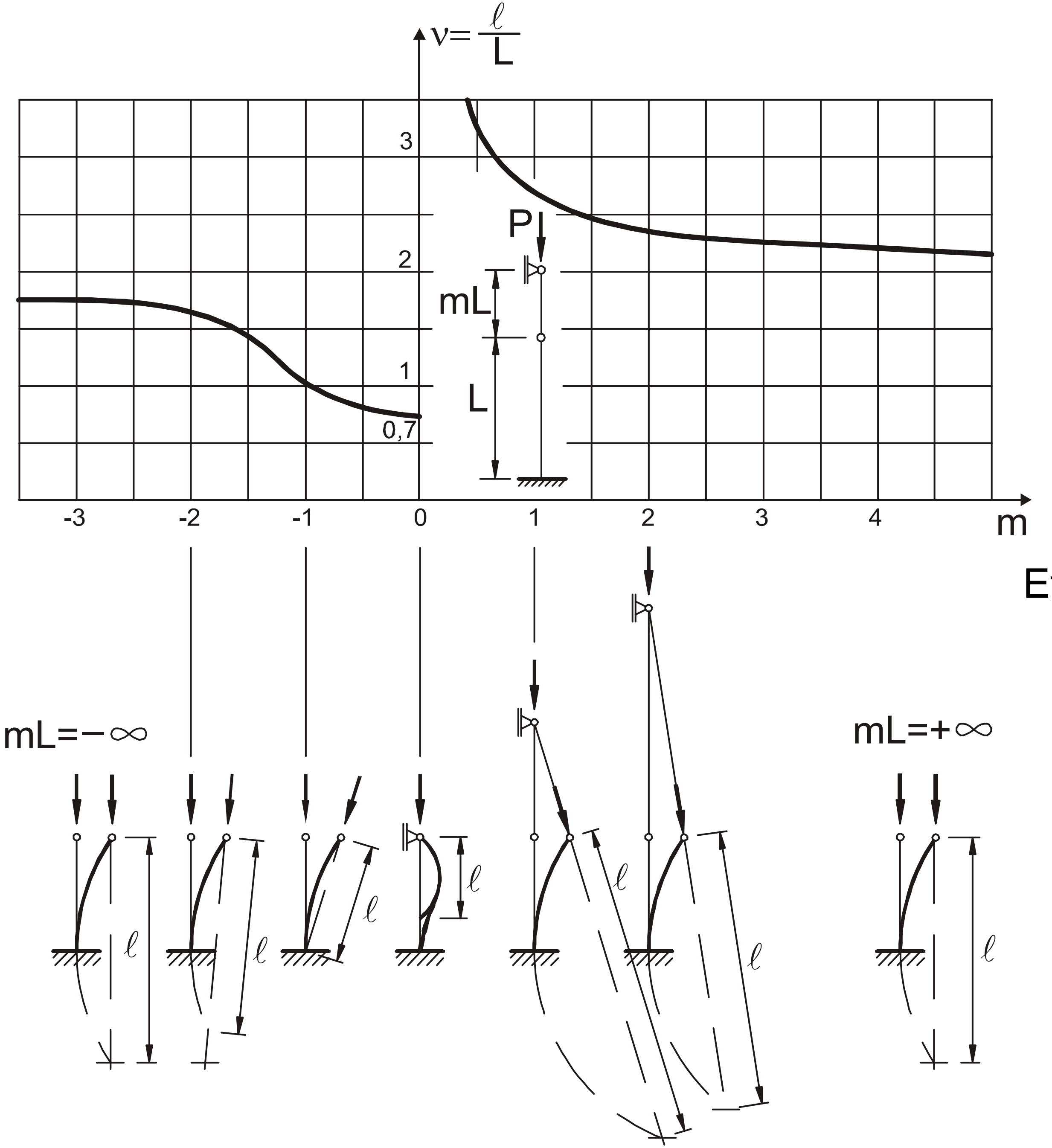
$$\det \begin{bmatrix} 0 & 1 & 1 + \frac{1}{m} \\ 1 & 0 & -\frac{1}{k \cdot m \cdot L} \\ \sin kL & \cos kL & 0 \end{bmatrix} = 0$$



$$\mathrm{tg}\, kL = (1 + m) \cdot k \cdot L$$

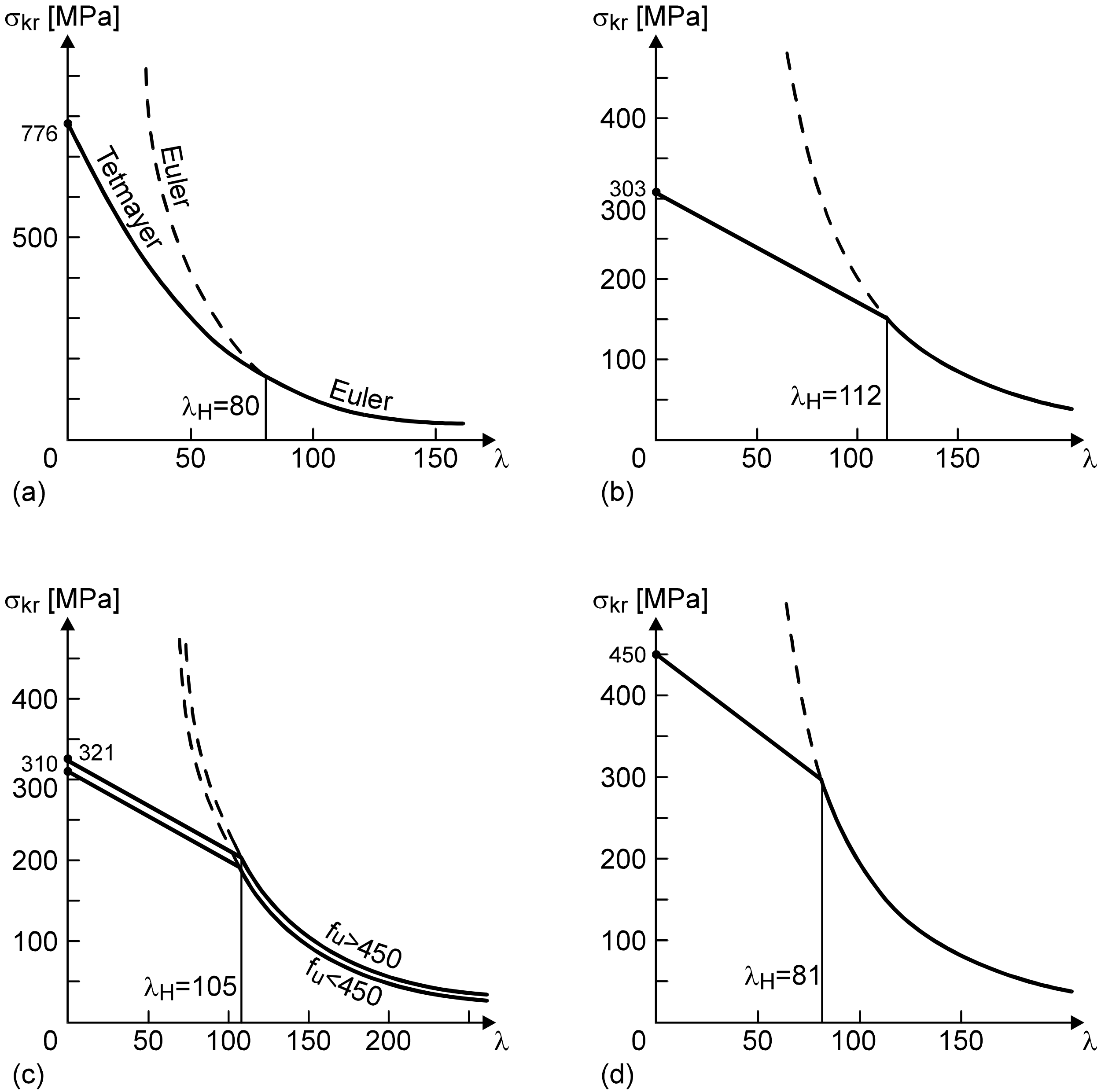
If $m = 0{,}1 \quad k \cdot L = 0.518$

$$P_{kr} = \frac{0.268 \cdot EI}{L^2}$$



Effective Length Factor with the Length of Pendelum

2.5. Inelastic Buckling of Columns



Tetmayer [1901]

(a) Cast iron

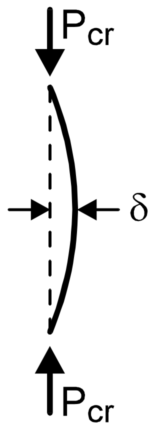
(b) Wrought iron

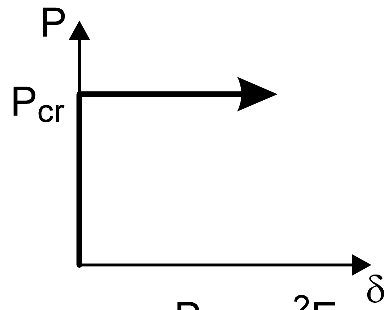
(c) Steel

(d) Ni-steel

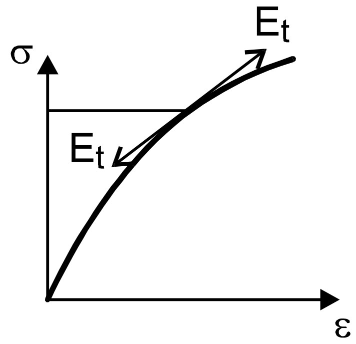
2.5.1 Early Development of Inelastic Column Theories

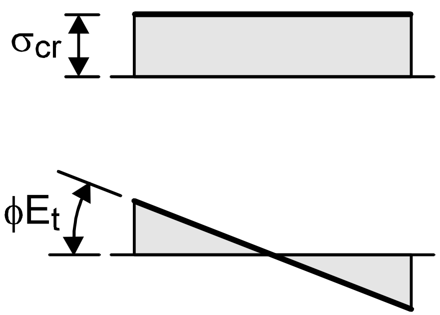
[Engesser, 1889]





$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E_t}{(L/i)^2}$$





$\sigma_{cr} = \frac{P_{cr}}{A}$: the stress in the cross-section at the tangent modulus load

additional bending stress ($\Sigma \sigma A = 0$)

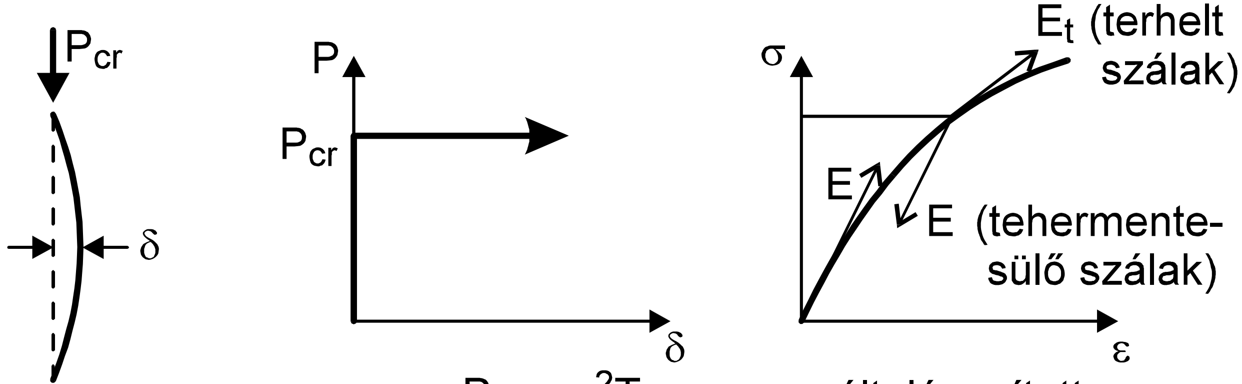
At the critical buckling load P_{cr} two positions of the column are possible, and these correspond to the stress distribution shown.

Note: the stresses in the cross-section at mid-height are critical

Original Engesser Theory

2.5.2 Reduced Modulus Theory

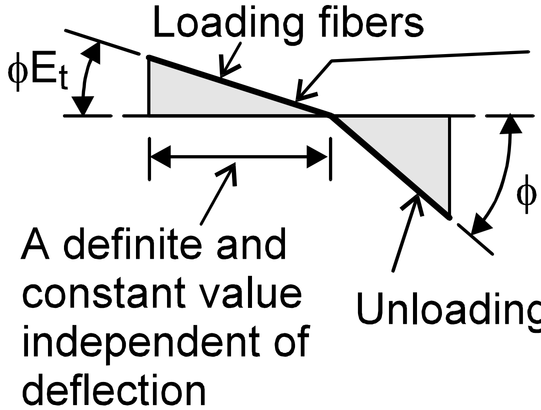
[Considére, 1891] [Jasinsky, 1895] [Engesser, 1898]



$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 T}{(L/i)^2}$$

általánosított σ–ε diagram

$\sigma_{cr} = \frac{P_{cr}}{A}$: the stress in the cross-section at the reduced modulus load



(Assumed as straight line. Actually a curve since E_t varies with σ)

Additional bending stress ($\Sigma\sigma A=0$)

$$\sigma_{kr} = \frac{\pi^2 \cdot T}{(L/i)^2}$$

$$M_{ext} = P \cdot y$$

$$M_{int} = \phi \cdot E_t \cdot d_1 \cdot \frac{b \cdot d_1}{2} \cdot \frac{2}{3} \cdot d_1 + \phi \cdot E \cdot d_2 \cdot \frac{b \cdot d_2}{2} \cdot \frac{2}{3} \cdot d_2 =$$

$$= \phi \cdot \frac{b}{3} \cdot (E_t \cdot d_1^3 + E \cdot d_2^3)$$

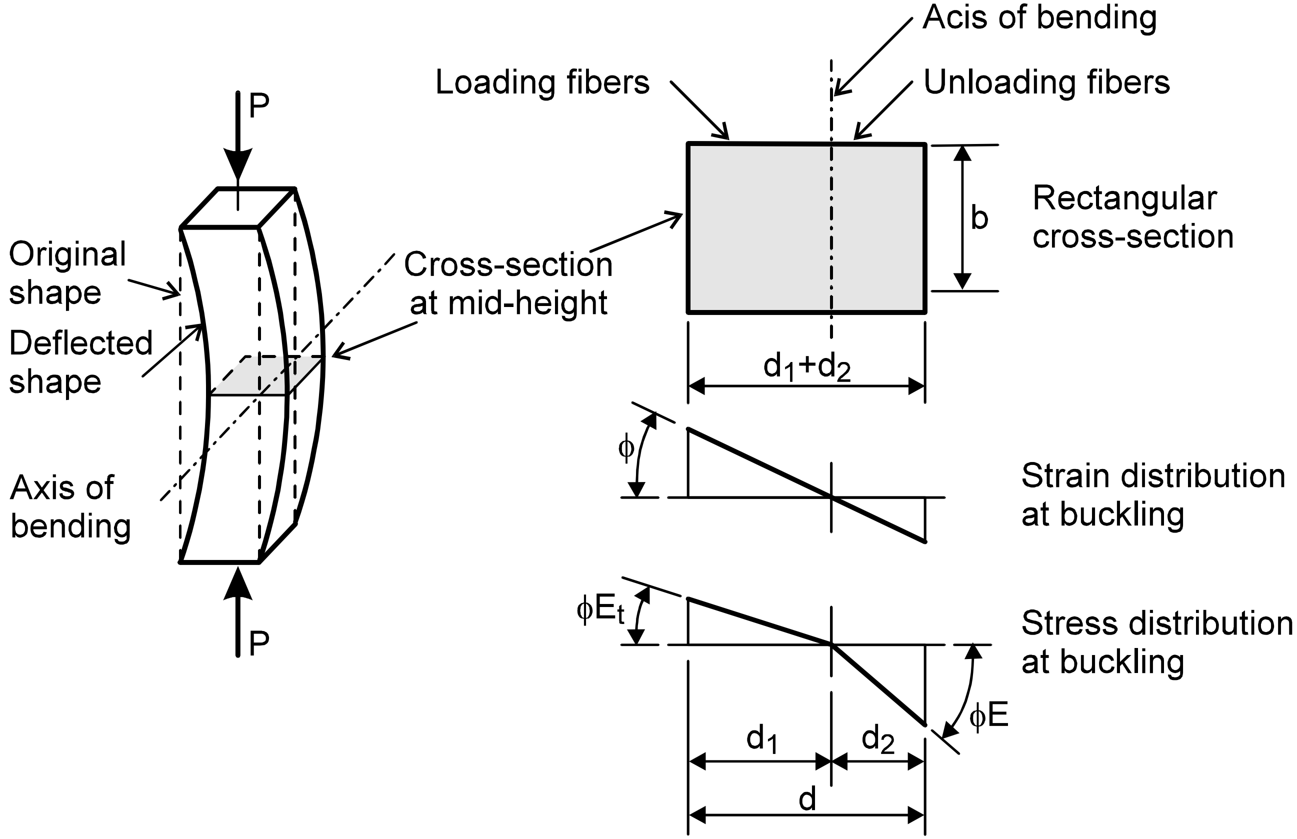
$$\phi = 1/\rho: \quad M_{int} = \frac{1}{\rho} \cdot \frac{b}{3} \cdot (E_t \cdot d_1^3 + E \cdot d_2^3) = \frac{T \cdot I}{\rho}$$

Reduced Modulus:

$$T = \frac{1}{I} \cdot \frac{b}{3} \cdot (E_t \cdot d_1^3 + E \cdot d_2^3)$$

$$\phi \cdot E_t \cdot d_1 \cdot \frac{b \cdot d_1}{2} = \phi \cdot E \cdot d_2 \cdot \frac{b \cdot d_2}{2}$$

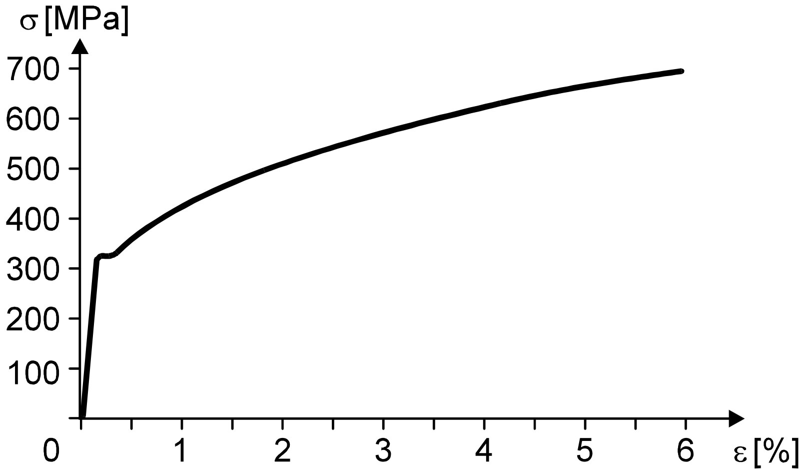
$$d_1^2 = \frac{E}{E_t} \cdot d_2^2$$



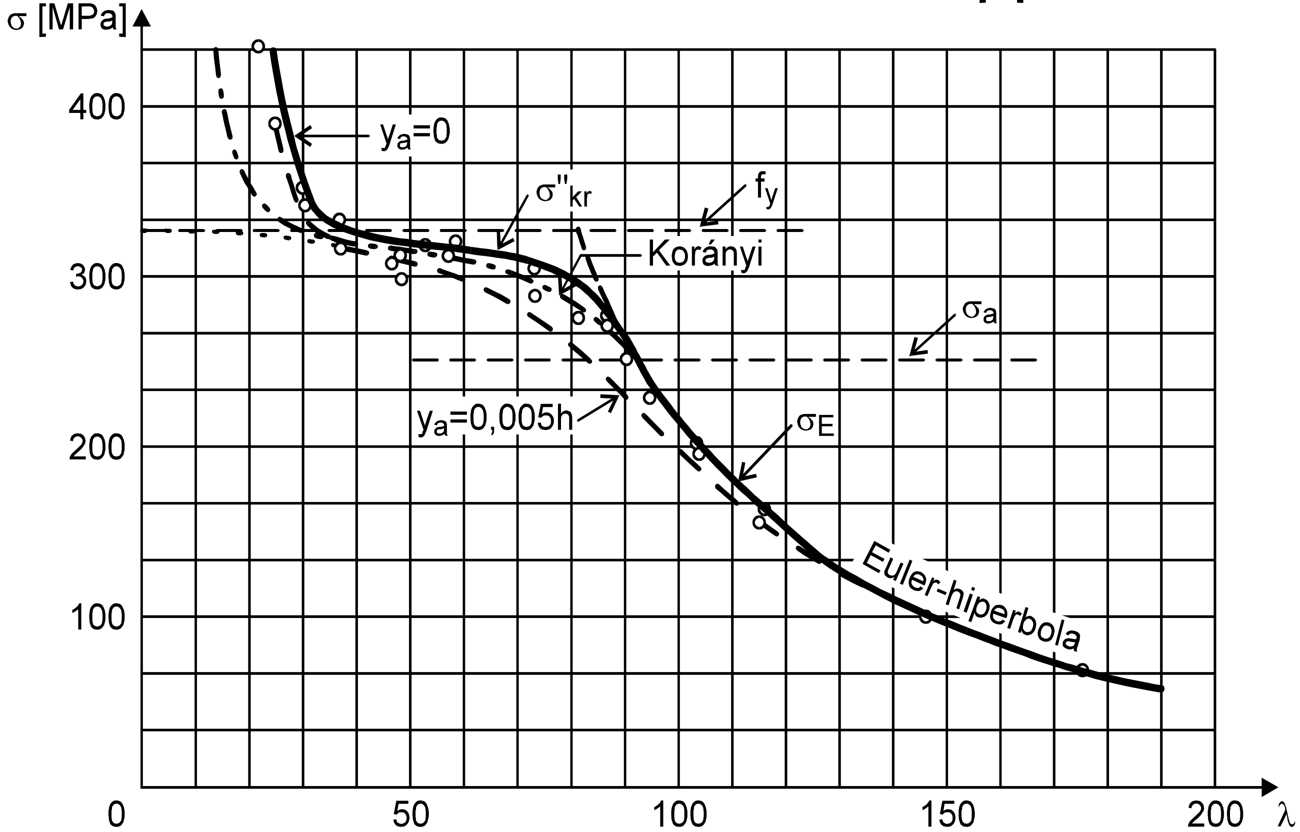
$$I = \frac{b \cdot (d_1 + d_2)^3}{12}$$

$$T = \frac{4E_t \cdot E}{\left(\sqrt{E_t} + \sqrt{E}\right)^2}$$

$$\boxed{E_t \le T \le E}$$

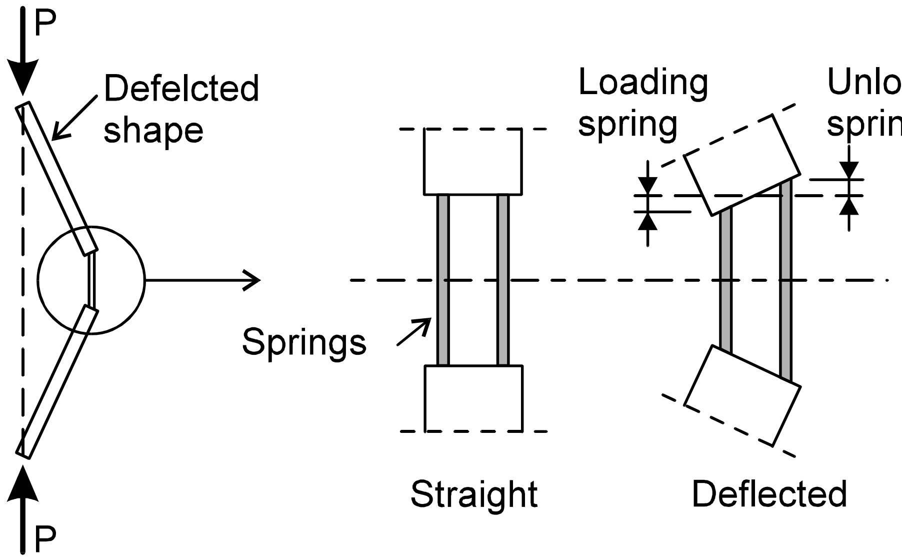


Tests by Karman

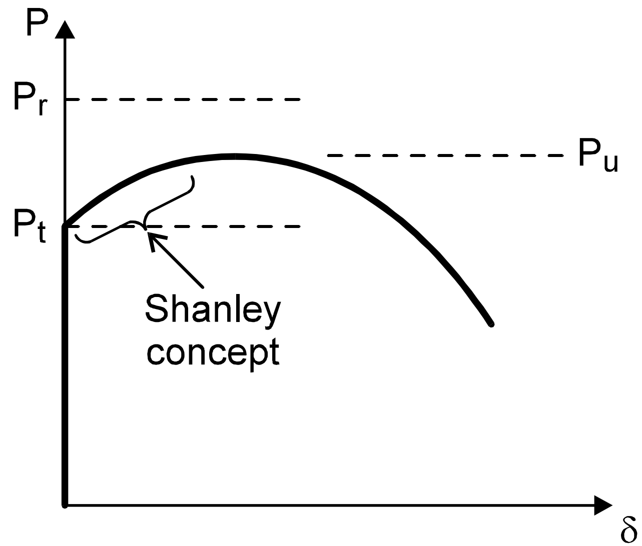


2.5.3 The Shanley Contribution

[Shanley, 1943] [Johnston, 1961] [Tall, 1964]

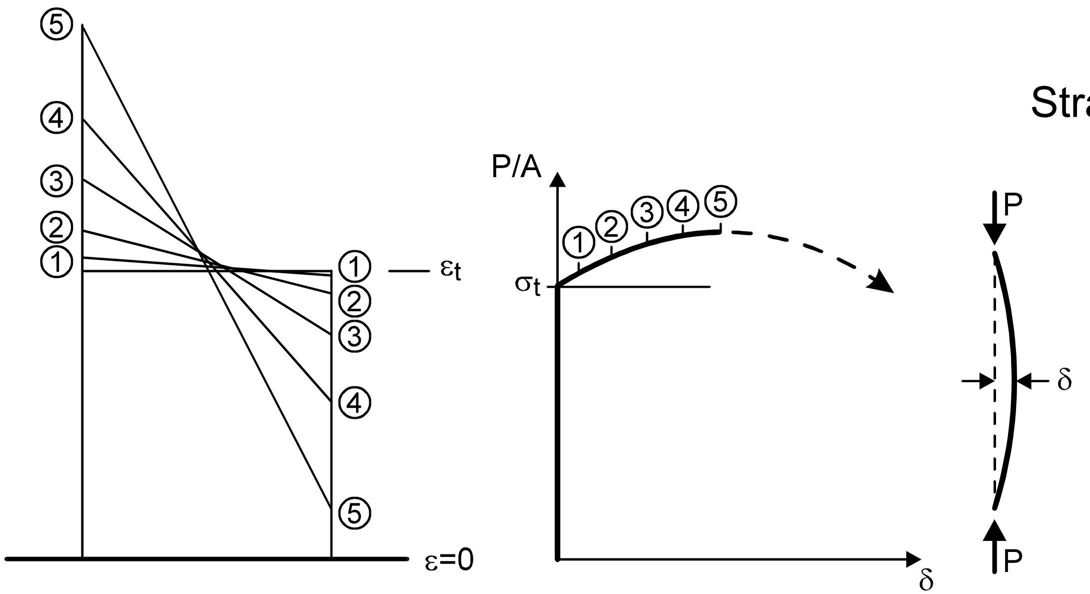


Shanley model column

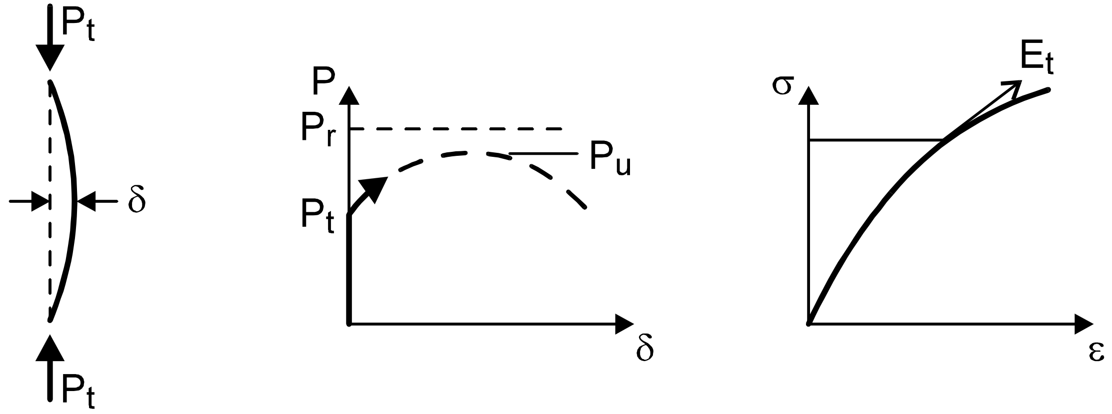


Deflection of initially straight centrally loaded column

Strain distribution and column deflection



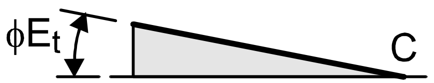
2.5.4 Tangent Modulus Theory



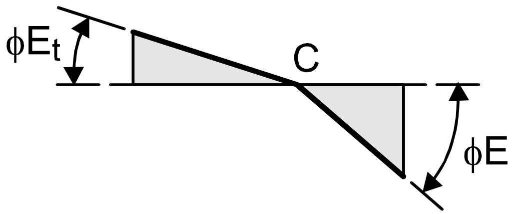
$$M_{\text{int}} = \frac{\phi \cdot E_t \cdot d}{2} \cdot \frac{b \cdot d}{4} \cdot \frac{2}{3} \cdot d = \phi \cdot E_t \cdot \frac{b \cdot d^3}{12} = \frac{E_t \cdot I}{\rho}$$

$\sigma_t = \frac{P_t}{A}$: the stress in the cross-section at the tangent modulus load

$$\sigma_t = \frac{\pi^2 \cdot E_t}{(L/i)^2}$$



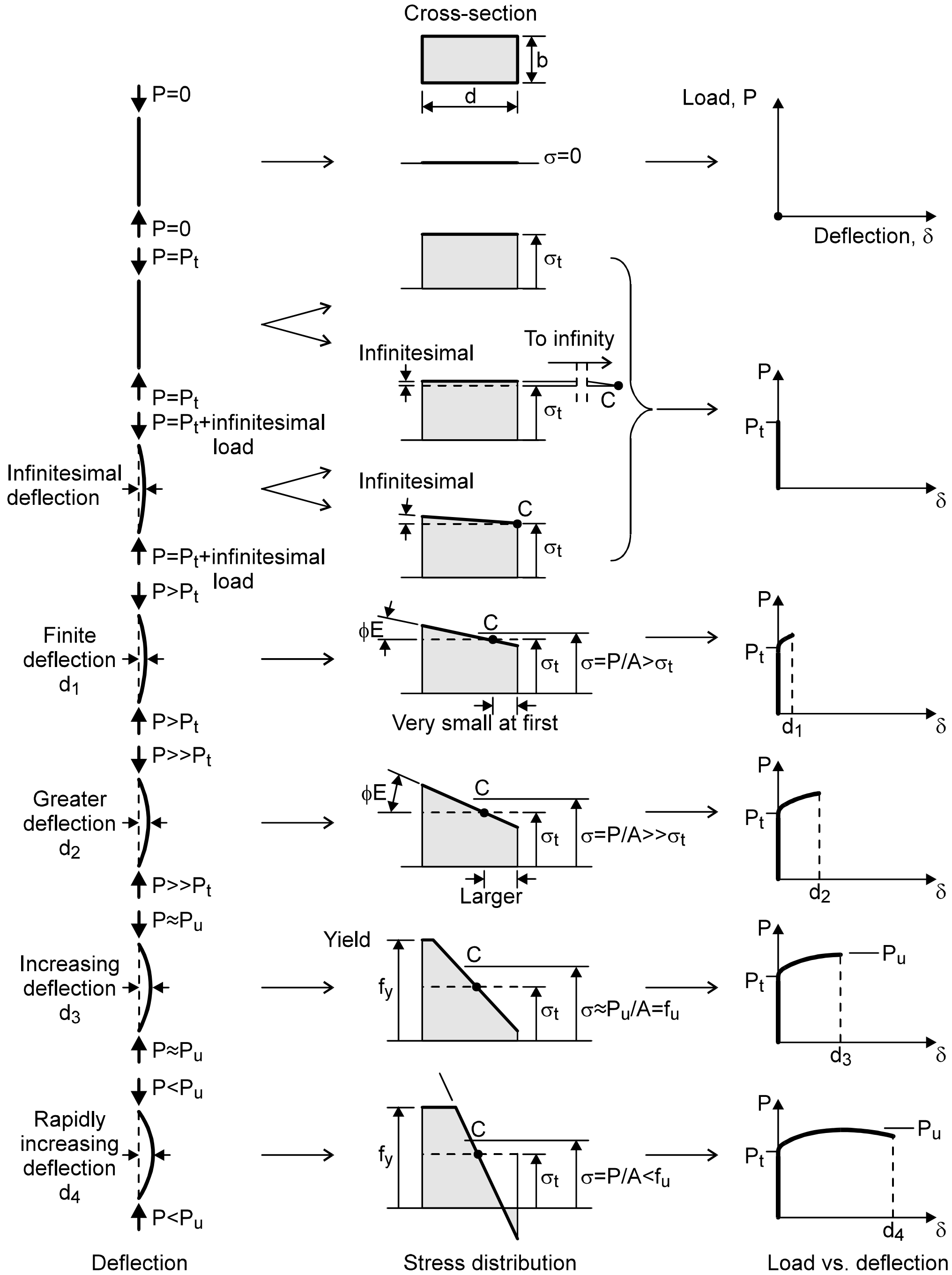
Additional infinitesimal bending stress at the tangent modulus load



Additional bending stress immediately above the tangent modulus load (Shanley contribution)

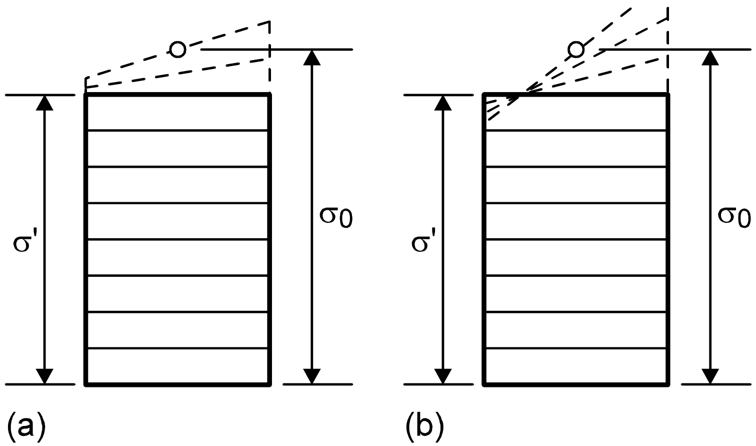
Tangent Modulus Concept

Progressive stress distribution as column is loaded

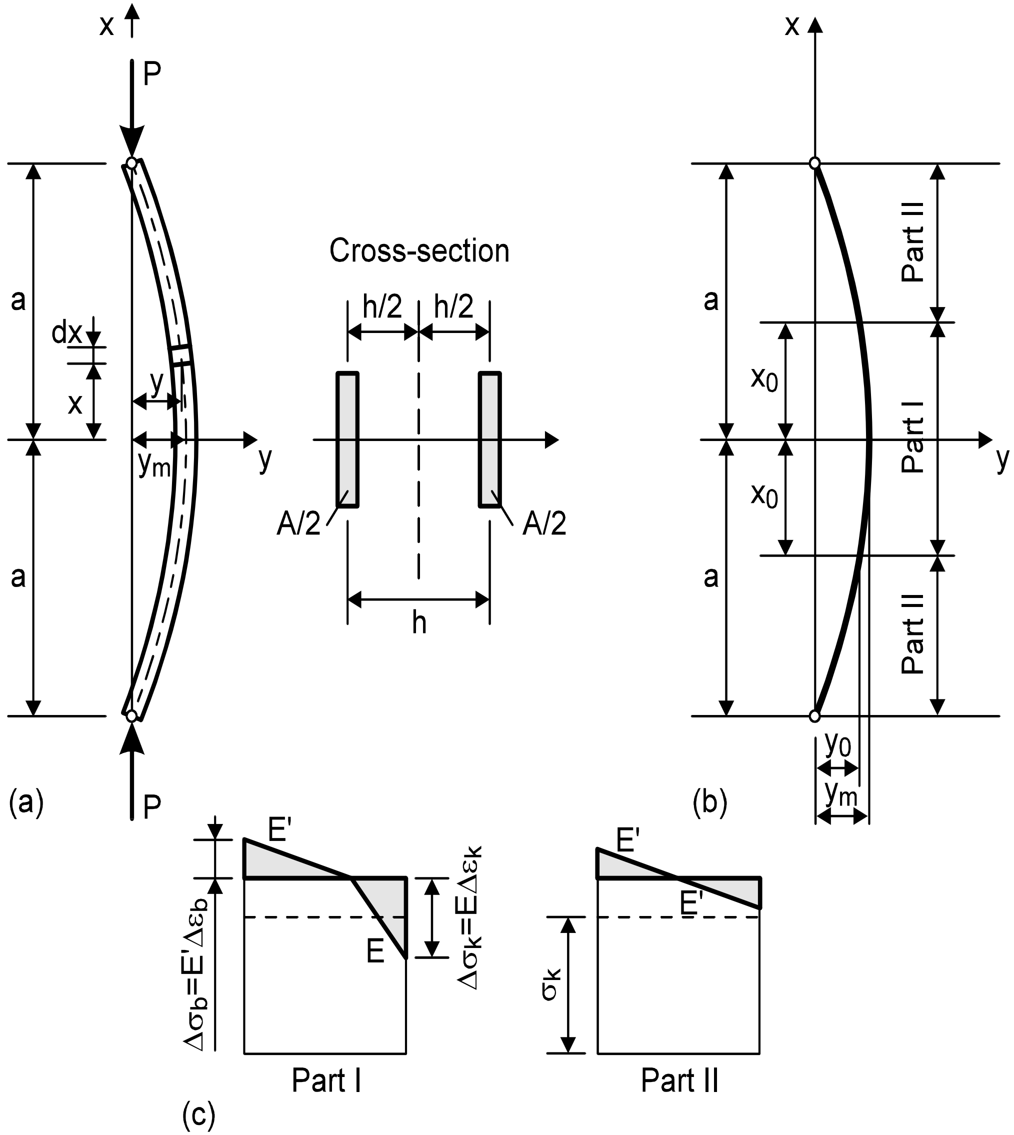


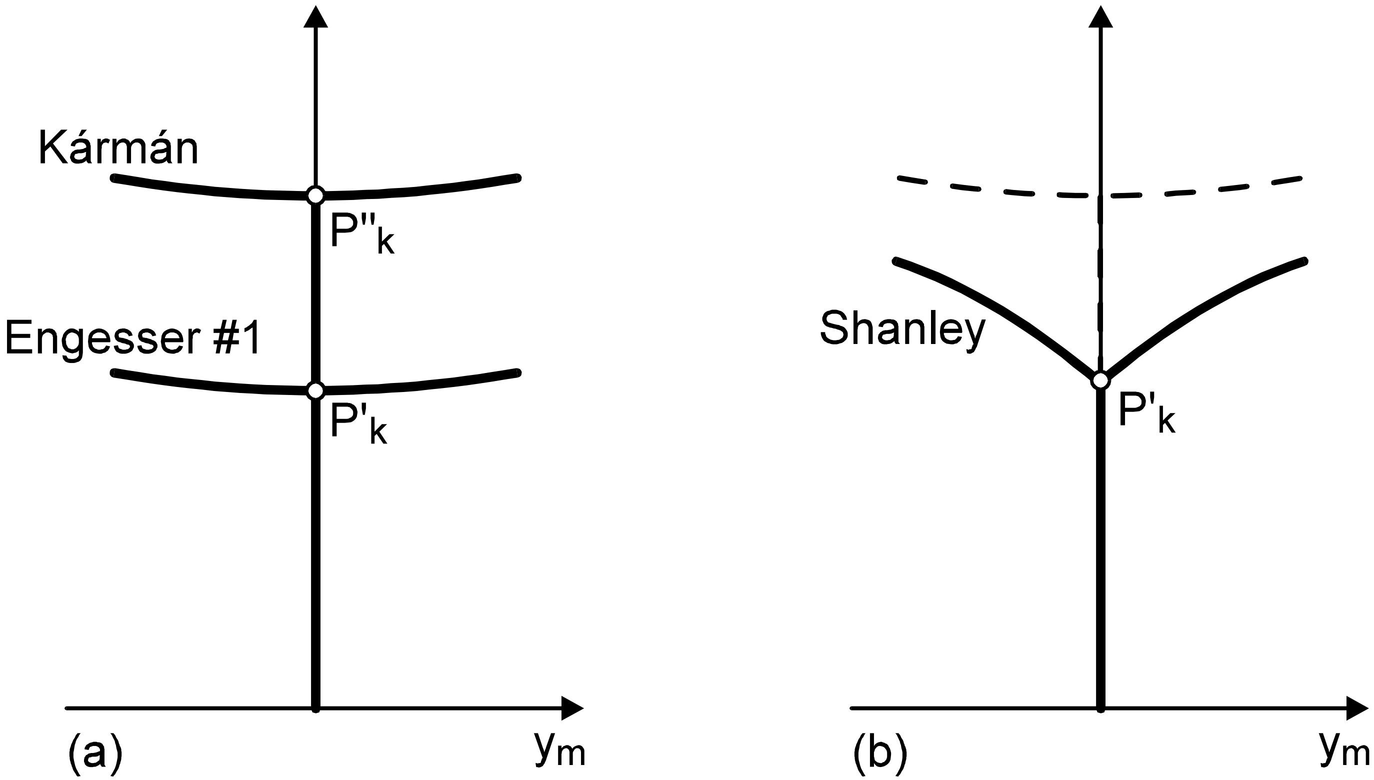
2.5.5 The Csonka Contribution

[Csonka, 1951]



Distribution of stresses





Buckling as of Engesser - Karman - Shanley

2.6. Critical Load of the Euler Column – Energy Method

2.6.1 Conservation of Energy Principle

A conservative system is in equilibrium if the strain energy stored is equal to the work performed by the external loads.

⇒ For an axially loaded bar it remains perfectly straight, the external work is given:

$$L_k = \frac{1}{2} P \cdot \Delta_a \qquad \Delta_a = \frac{P \cdot L}{EA}$$

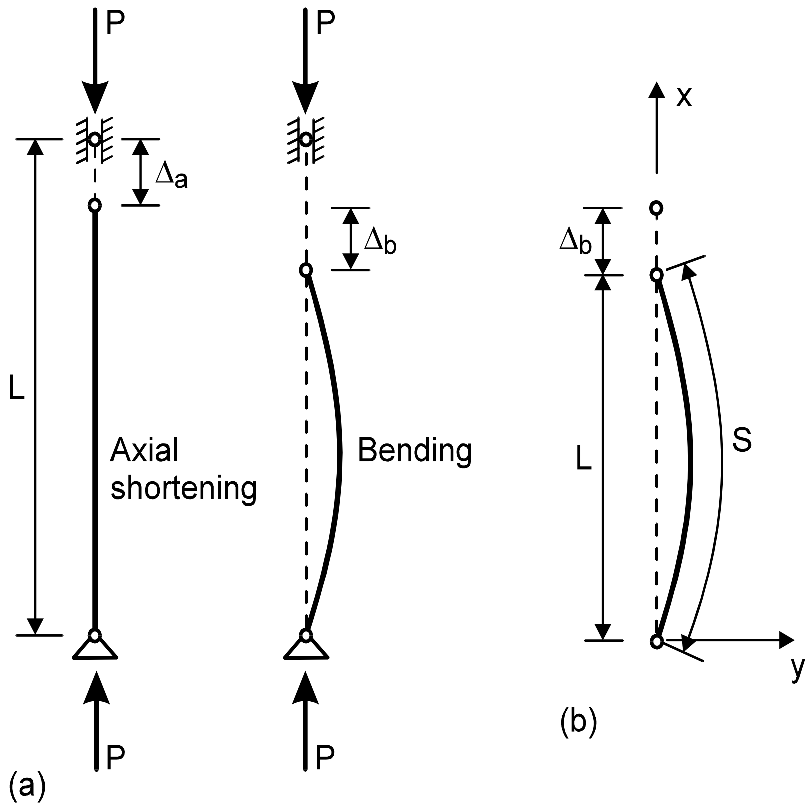
Strain energy stored in the member:

$$L_b = \frac{P^2 \cdot L}{2EA} \qquad \frac{P^2 \cdot L}{2EA} \equiv \frac{P^2 \cdot L}{2EA}$$

$$\boxed{L_k = L_b} \qquad \Delta_b = S - L$$

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \cdot \mathrm{d}x$$

$$S = \int_0^L \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$$



Column shortening due to axial compression and bending

Binomial theorem:

$$(a+b)^n = a^n + \frac{n}{1!}\cdot a^{n-1}\cdot b + \frac{n\cdot(n-1)}{2!}\cdot a^{n-2}\cdot b^2 + \dots$$

If deformations are assumed to be small.

$$S = \int_0^L \left[1 + \frac{1}{2}\cdot\left(\frac{\mathrm{d}\,y}{\mathrm{d}\,x}\right)^2\right]\mathrm{d}\,x$$

$$\Delta_b = S - L = \frac{1}{2}\cdot\int_0^L \left(\frac{\mathrm{d}\,y}{\mathrm{d}\,x}\right)^2 \mathrm{d}\,x$$

External work:

$$L_k = P\cdot\Delta_b = \frac{P}{2}\cdot\int_0^L \left(\frac{\mathrm{d}\,y}{\mathrm{d}\,x}\right)^2 \mathrm{d}\,x$$

Strain energy:

$$L_b = \frac{1}{2}\cdot\int_0^L \frac{M^2}{EI}\mathrm{d}\,x = \frac{1}{2}\cdot\int \frac{P_{kr}^2\cdot y^2}{EI}\mathrm{d}\,x = \frac{1}{2}\cdot P_{kr}^2\cdot\int_0^L \frac{y^2}{EI}\mathrm{d}\,x$$

$$L_b = \frac{1}{2}\cdot\int_0^L M\cdot\frac{1}{\rho}\mathrm{d}\,x \qquad M = \frac{EI}{\rho} \qquad \frac{1}{\rho} \cong -\frac{\mathrm{d}^2\,y}{\mathrm{d}\,x^2}$$

$$L_b = \frac{EI}{2}\cdot\int_0^L \left(\frac{\mathrm{d}^2\,y}{\mathrm{d}\,x^2}\right)^2 \mathrm{d}\,x$$

Grammel-quotient:

$$P_{cr,G} = \frac{\int_0^L y'^2\,\mathrm{d}\,x}{\int_0^L \frac{y^2}{EI}\mathrm{d}\,x}$$

Rayleigh-quotient:

$$P_{cr,R} = \frac{\int_0^L EI\cdot y''^2\,\mathrm{d}\,x}{\int_0^L y'^2\,\mathrm{d}\,x}$$

$$y = A\cdot\sin\frac{\pi x}{L}$$

$$L_k = \frac{A^2\cdot P\cdot\pi^2}{2L^2}\cdot\int_0^L \cos^2\frac{\pi x}{L}\mathrm{d}\,x = \frac{A^2\cdot P\cdot\pi^2}{4L}$$

$$L_b = \frac{A^2\cdot EI\cdot\pi^4}{2L^4}\cdot\int_0^L \sin^2\frac{\pi x}{L}\mathrm{d}\,x = \frac{A^2\cdot EI\cdot\pi^4}{4L^3}$$

$$\boxed{L_k = L_b} \longrightarrow \boxed{P_{kr} = \frac{\pi^2\cdot EI}{L^2}}$$

2.6.2 Calculus of Variations

The calculus of variations is a generalisation of the max. or min. problem of ordinary calculus. It seeks to determine a function y=y(x) that extremizes a definite integral:

$$I = \int_{x_1}^{x_2} \mathrm{f}(x, y, y', \ldots, y^{(n)}) \,\mathrm{d}\, x = \mathrm{extr!}$$

Thus the calculus of variations is not a computational tool for solving a problem. It is only a device for obtaining the governing equations of the problem.

[Hoff, 1956] [Chajes, 1974]

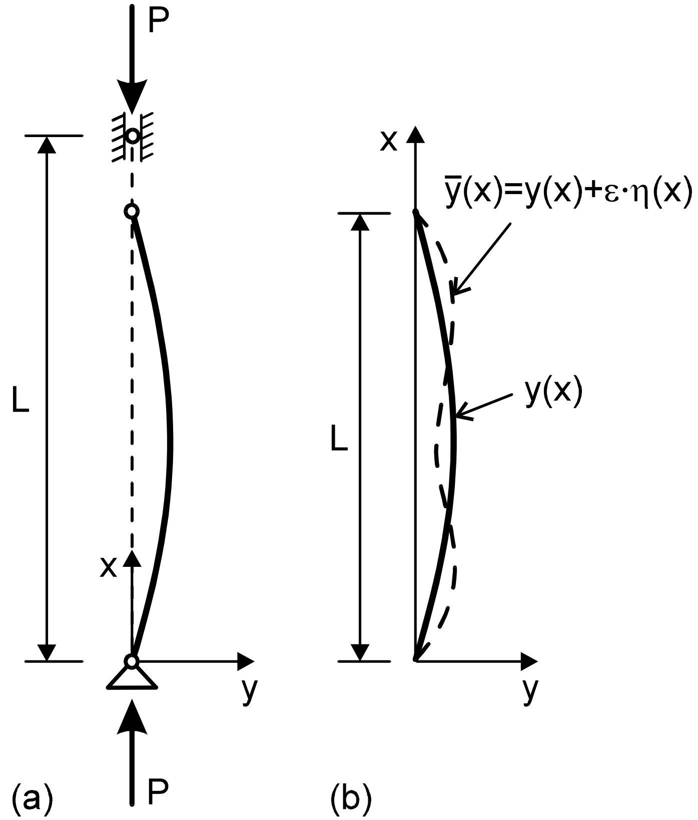
Strain energy: $L_b = \int_0^L \frac{EI}{2} \cdot (y'')^2 \,\mathrm{d}\, x$

External work: $L_k = -\int_0^L \frac{P}{2} \cdot (y')^2 \,\mathrm{d}\, x$

Potential energy: $\delta^2 \Pi = \int_0^L \left[\frac{EI}{2} \cdot (y'')^2 - \frac{P}{2} (y')^2 \right] \mathrm{d}\, x$

$$\delta(\delta^2 \Pi_0) = 0$$

Boundary condition: $y(x = 0) = y(x = L) = 0$



Nearly function: $\bar{y}(x) = y(x) + \varepsilon \cdot \eta(x)$

$$\delta^2 \Pi_0 = \int_0^L \left[\frac{EI}{2} \cdot (y'' + \varepsilon \cdot \eta'')^2 - \frac{P}{2} \cdot (y' + \varepsilon \cdot \eta')^2 \right] \mathrm{d}\, x$$

Extremum value: $\left. \frac{\mathrm{d}(\delta^2 \Pi)}{\mathrm{d}\,\varepsilon} \right|_{\varepsilon=0} = 0$

$$\frac{\mathrm{d}(\delta^2 \Pi)}{\mathrm{d}\varepsilon} = \int_0^L [EI \cdot (y'' + \varepsilon \cdot \eta'') \cdot \eta'' - P \cdot (y' + \varepsilon \cdot \eta') \cdot \eta'] \mathrm{d}x$$

$\varepsilon = 0:$ $\int_0^L (EI \cdot y'' \cdot \eta'' - P \cdot y' \cdot \eta') \,\mathrm{d}\, x = 0$

Second term: $\int_0^L y' \cdot \eta' \,\mathrm{d}\, x = -\int_0^L \eta \cdot y'' \,\mathrm{d}\, x$

First term: $\int_0^L y'' \cdot \eta'' \cdot \mathrm{d}x = [y'' \cdot \eta']_{x=0}^{L} + \int_0^L \eta \cdot y^{IV} \,\mathrm{d}\, x$

$$\int_0^L (EI \cdot y^{IV} + P \cdot y'') \cdot \eta \,\mathrm{d}\, x + [EI \cdot y'' \cdot \eta']_{x=0}^{L} = 0$$

$$\int_0^L (EI \cdot y^{IV} + P \cdot y'') \cdot \eta \,\mathrm{d}\, x = 0$$

$$[EI \cdot y'' \cdot \eta']_{x=0}^{L} = 0$$

$\eta'(x=0) \neq 0$ $\eta'(x=L) \neq 0$

$\eta(x) \not\equiv 0$ $\eta'(x=0)$

$\eta'(x=L)$

$$EI \cdot y^{IV} + P \cdot y'' = 0$$

Natural boundary condition:

$\left. EI \cdot y'' \right|_{x=0} = 0$ $\left. EI \cdot y'' \right|_{x=L} = 0$

Geometric boundary condition:

$y(x=0) = 0$ $y(x=L) = 0$

2.6.3 Buckling Load of Column with Variable Cross-section
Rayleigh–Ritz Method

Buckled shape: $y = a \cdot \sin \frac{\pi x}{L}$ [Koranyi, 1965]

Strain energy:

$$L_b = 2 \cdot \left[\frac{EI_0}{8} \cdot \int_0^{L/4} (y'')^2 \,\mathrm{d}x + \frac{EI_0}{2} \cdot \int_{L/4}^{L/2} (y'')^2 \,\mathrm{d}x \right]$$

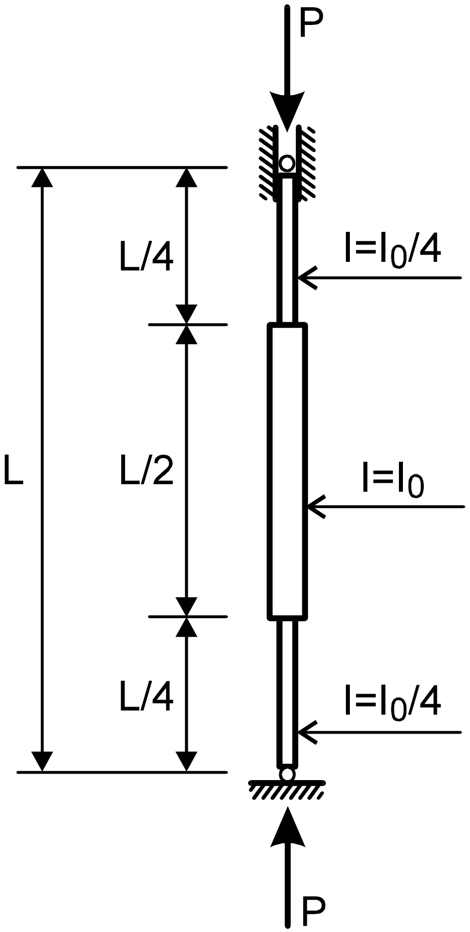
$$\int_0^{L/4} (y'')^2 \,\mathrm{d}x = \frac{a^2 \cdot \pi^4}{L^4} \cdot \int_0^{L/4} \sin^2 \frac{\pi x}{L} \,\mathrm{d}x = \frac{0{,}045 a^2 \cdot \pi^4}{L^3}$$

$$\int_{L/4}^{L/2} (y'')^2 \,\mathrm{d}x = \frac{a^2 \cdot \pi^4}{L^4} \cdot \int_{L/4}^{L/2} \sin^2 \frac{\pi x}{L} \,\mathrm{d}x = \frac{0{,}205 a^2 \cdot \pi^4}{L^3}$$

$$L_b = 0{,}216 \cdot \frac{EI_0 \cdot a^2 \cdot \pi^4}{L^3}$$

External work:

$$L_k = -\frac{P}{2} \cdot \int_0^{L} (y')^2 \,\mathrm{d}x = -\frac{P \cdot a^2 \cdot \pi^2}{2L^2} \cdot \int_0^{L} \cos^2 \frac{\pi x}{L} \,\mathrm{d}x = -\frac{P \cdot a^2 \cdot \pi^2}{4L}$$



Column with varying moment of inertia

$$\delta^2\Pi = 0{,}216 \cdot \frac{EI_0 \cdot a^2 \cdot \pi^4}{L^3} - \frac{P \cdot a^2 \cdot \pi^2}{4L}$$

$$\frac{\mathrm{d}(\delta^2\Pi)}{\mathrm{d}\,a} = 0{,}432 \cdot \frac{EI_0 \cdot a \cdot \pi^4}{L^3} - \frac{P \cdot a \cdot \pi^2}{2L} = 0$$

$$a \cdot \left(0{,}864 \cdot \frac{EI_0 \cdot \pi^2}{L^2} - P \right) = 0$$

Critical load:

$$P_{kr} = 0{,}864 \cdot \frac{\pi^2 \cdot EI_0}{L^2}$$

Exact answer [Timoshenko, Gere, 1961]:

$$P_{kr} = 0{,}65 \cdot \frac{\pi^2 \cdot EI_0}{L^2}$$

Error is about 33%.

If deflection curve is:

$$y = a_1 \cdot \sin\frac{\pi x}{L} + a_2 \cdot \sin\frac{3\pi x}{L}$$

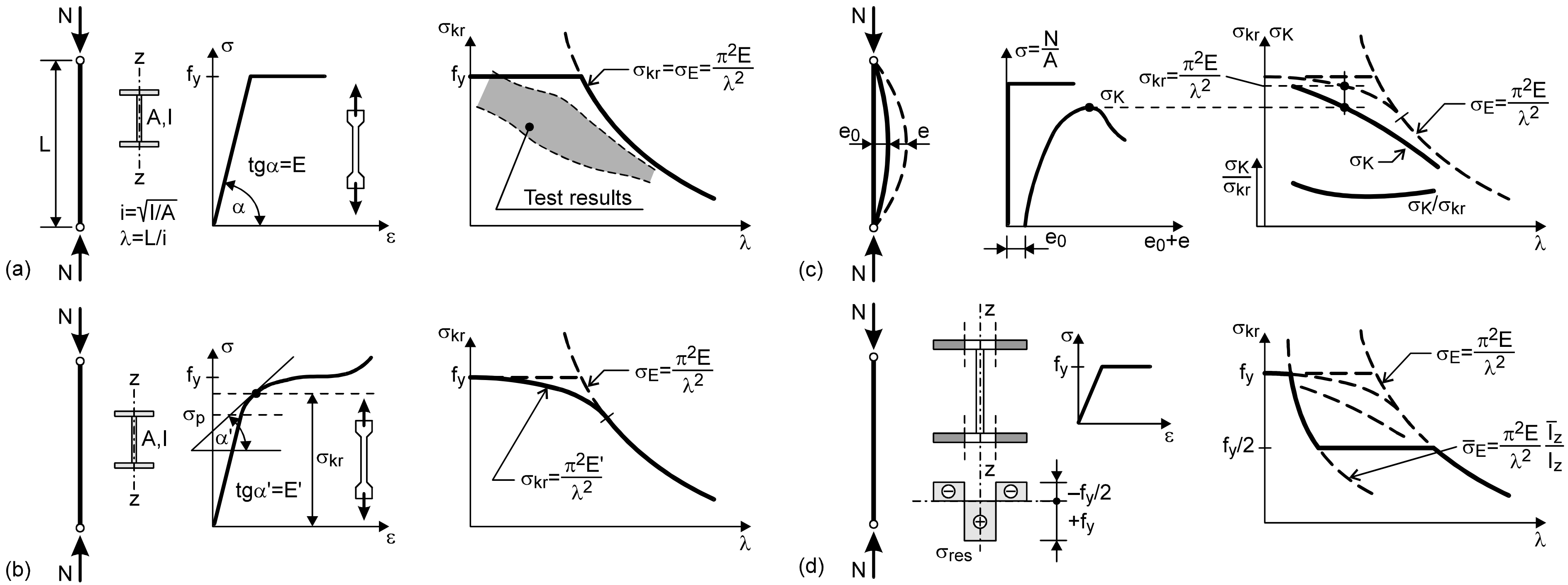
Critical load:

$$P_{kr} = 0{,}735 \cdot \frac{\pi^2 \cdot EI_0}{L^2}$$

Error is about 13%.

2.7. Design of Columns

2.7.1 Historical Background



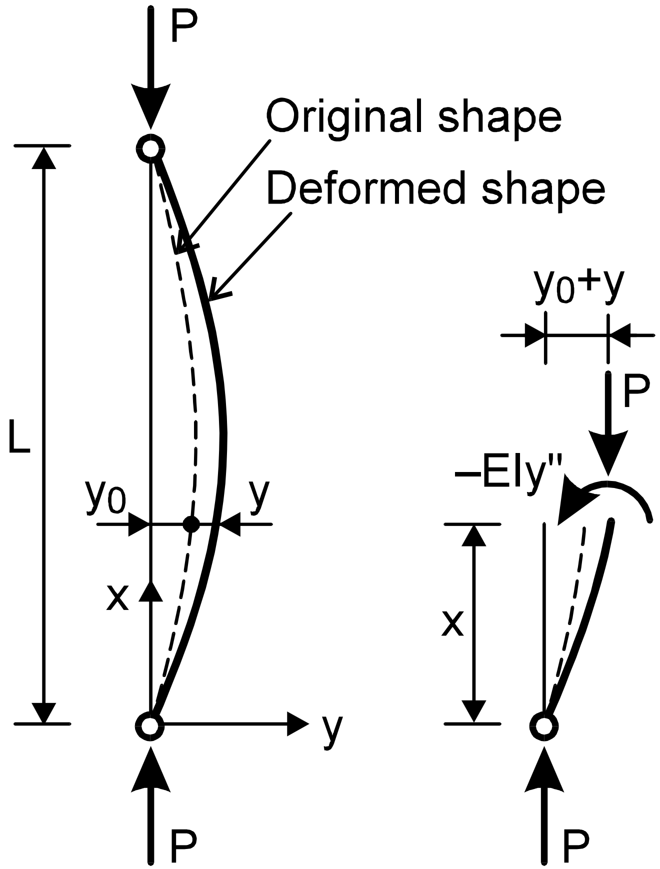
2.7.2 Ayrton–Perry formula (1886)

[Rondal, Maquoi, 1979]

Initial deformation: $y_0 = e_0 \cdot \sin \frac{\pi x}{L}$

Deflection at midspan: $e = e_0 \cdot \frac{1}{1-\alpha} = e_0 \cdot \frac{1}{1 - P/P_E}$

First yield limit state: $\frac{P}{P_y} + \frac{M}{M_y} = 1$



$$M = P \cdot e = P \cdot \frac{e_0}{1 - P/P_E}$$

$$\frac{P}{P_y} + \frac{P \cdot e_0}{\left(1 - \frac{P}{P_E}\right) \cdot M_y} = 1$$

$$P_y = A \cdot \sigma_H$$

$$M_y = W \cdot \sigma_H$$

$$\frac{P}{P_y} + \frac{P \cdot e_0}{\left(1 - \frac{P}{P_y} \cdot \frac{P_y}{P_E}\right) \cdot M_y} = 1$$

$$P_E = \frac{\pi^2 \cdot EI}{L^2} = \frac{\pi^2 \cdot EA}{\lambda^2}$$

$$\lambda = \bar{\lambda} \cdot \lambda_E = \bar{\lambda} \cdot \pi \cdot \sqrt{\frac{E}{\sigma_H}} \qquad \frac{P_y}{P_E} = \bar{\lambda}^2$$

$$\eta = e_0 \cdot \frac{A}{W}$$

$$\frac{P}{P_y} + \frac{\frac{P}{P_y}}{1 - \frac{P}{P_y}\bar{\lambda}^2} \cdot \eta = 1$$

$$\varphi = P / P_y \quad \rightarrow \quad \varphi + \frac{\varphi}{1 - \varphi \cdot \bar{\lambda}^2} \cdot \eta = 1$$

$$(1-\varphi) \cdot (1 - \varphi \cdot \bar{\lambda}^2) = \eta \cdot \varphi$$

$$\bar{\lambda}^2 \cdot \varphi^2 - (1 + \eta + \bar{\lambda}^2) \cdot \varphi + 1 = 0$$

$$\boxed{\varphi = \frac{(1 + \eta + \bar{\lambda}^2) - \sqrt{(1 + \eta + \bar{\lambda}^2)^2 - 4\bar{\lambda}^2}}{2\bar{\lambda}^2}}$$

[Robertson, 1925]: $\eta = 0{,}003 \cdot \lambda$

$$\eta = b_0 \cdot A / W_z \qquad b_0 = L / \gamma$$

$$W_z = I_z / v \qquad \lambda = L / i_z$$

$$\eta = \frac{\lambda}{\gamma \cdot i_z / v}$$

[Dutheil, 1947]: $\eta = \dfrac{C}{\pi^2 \cdot E} \cdot f_y \cdot \lambda^2$

$$C = \frac{1}{12} \qquad \eta = C \cdot \bar{\lambda}^2$$

[Dwight, 1972]: $\eta = \alpha \cdot (\lambda - \lambda_0)$

[Rondal, Maquoi, 1978]:

$$\eta = \alpha \cdot (\bar{\lambda} - \bar{\lambda}_0)$$

$$\eta = \alpha \cdot \sqrt{\bar{\lambda}^2 - \bar{\lambda}_0^2}$$