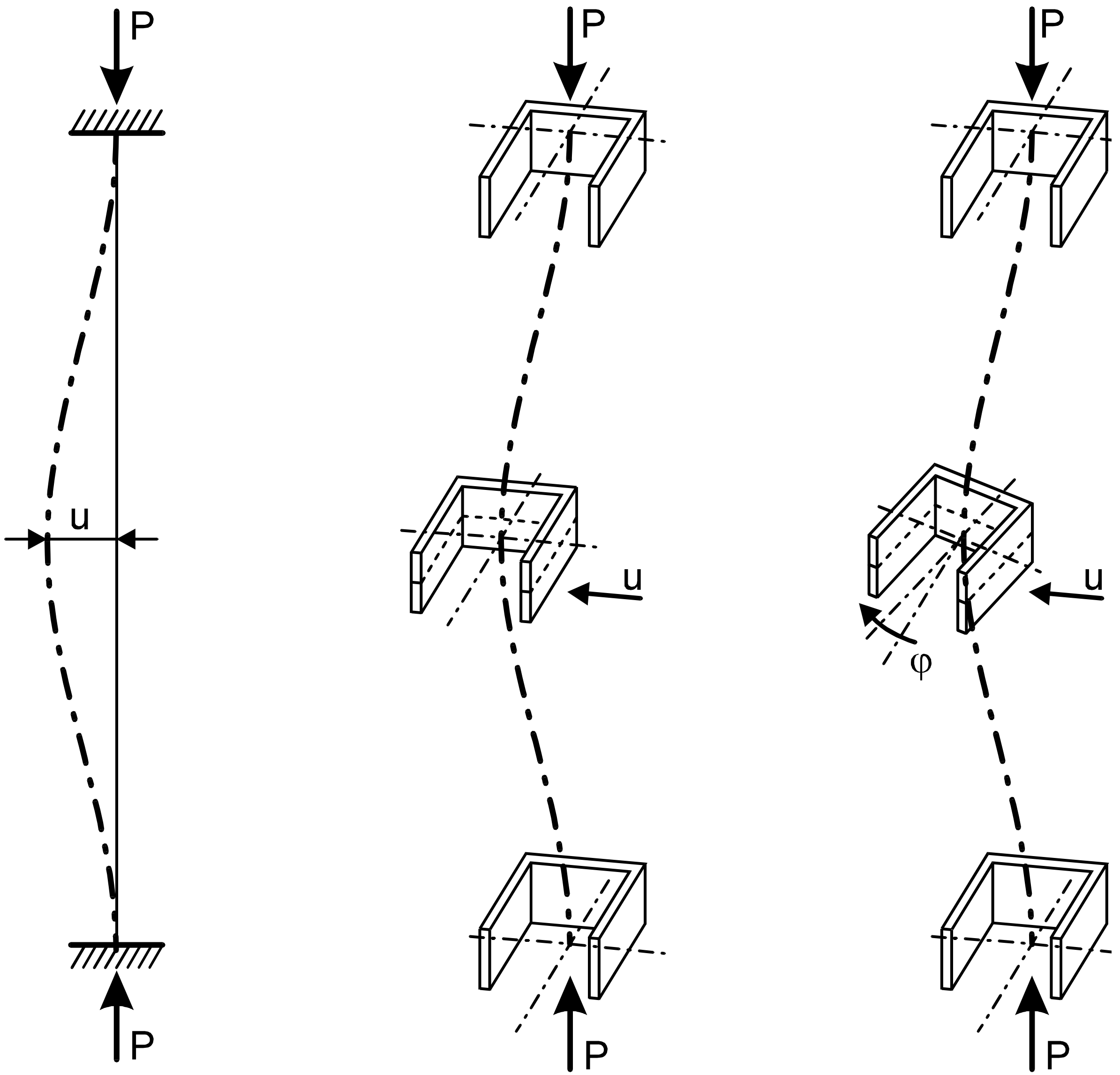
Készült az ERFP – DD2002 – HU – B – 01 szerződésszámú projekt támogatásával

# Chapter 5

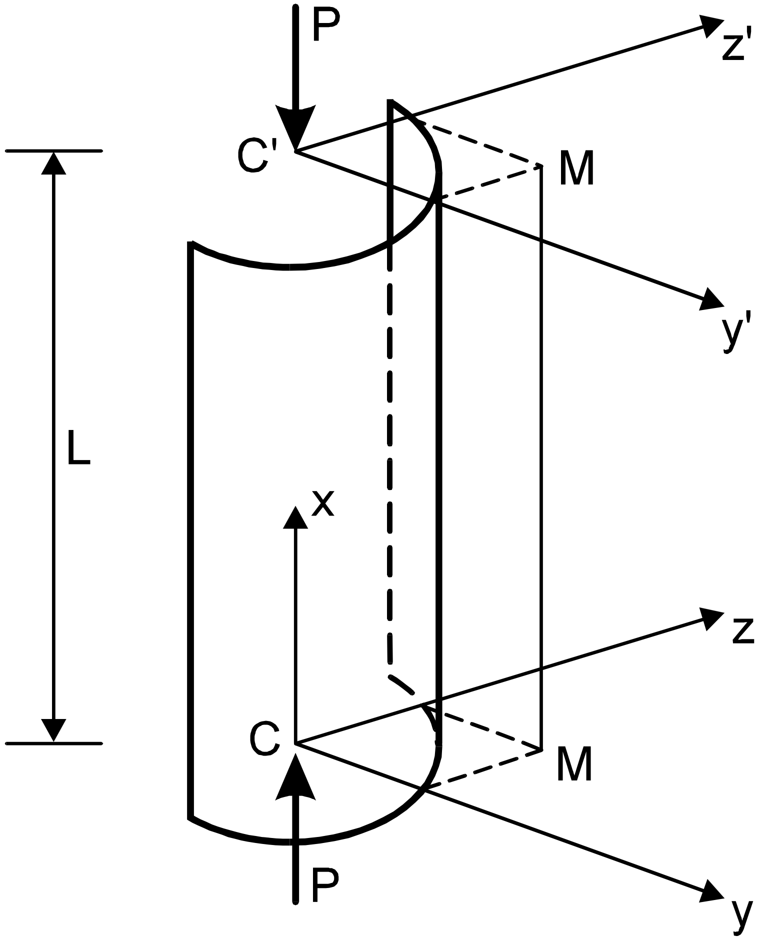
# Spatial Buckling of Struts

# 5.1. Lateral-Torsional Buckling of Columns

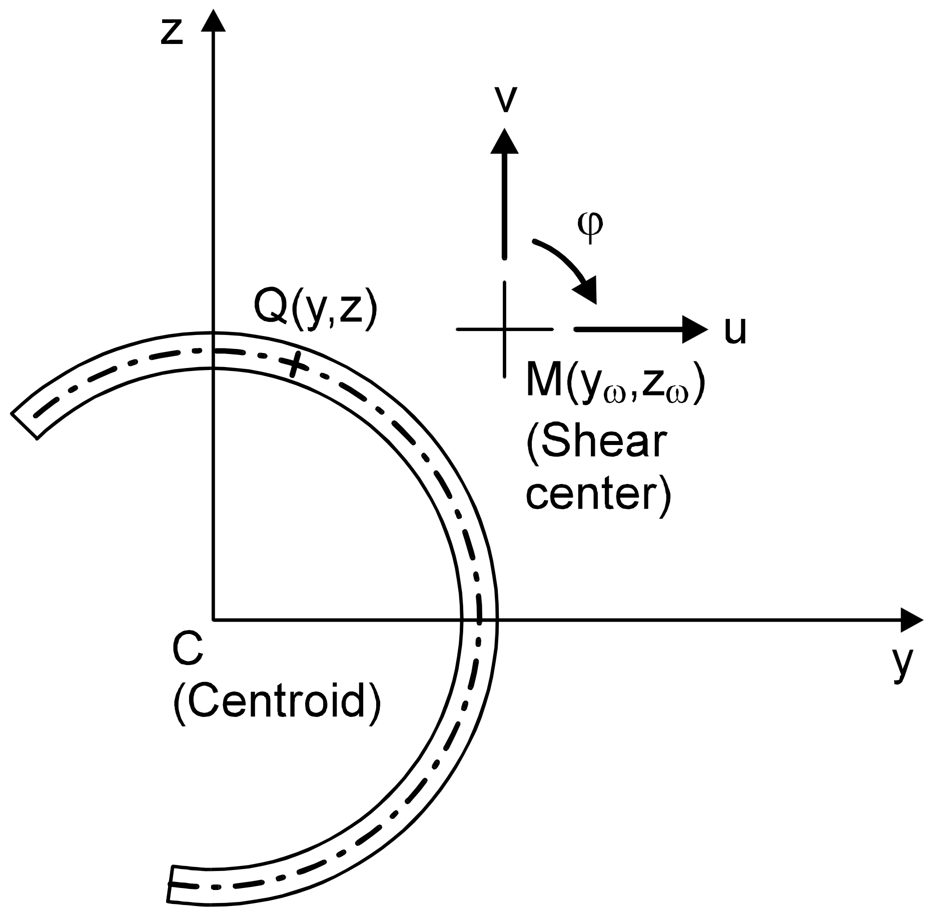



[Halasz, 1965]
[Wagner, 1936]
[Kappus, 1937]
[Goodier, 1941]
[Bleich, 1952]
[Timoshenko, Gere, 1961]
[Vlasov, 1961]
[Murray, 1984]

# 5.1.1 Equilibrium Method for General Open Cross-section Column




$(u, v, \varphi)$: Displacement of shear center M



## (a) Displacements of the cross-section

$$\overline{QM} = a$$

$$u_Q = u + a \cdot \varphi \cdot \sin\alpha$$

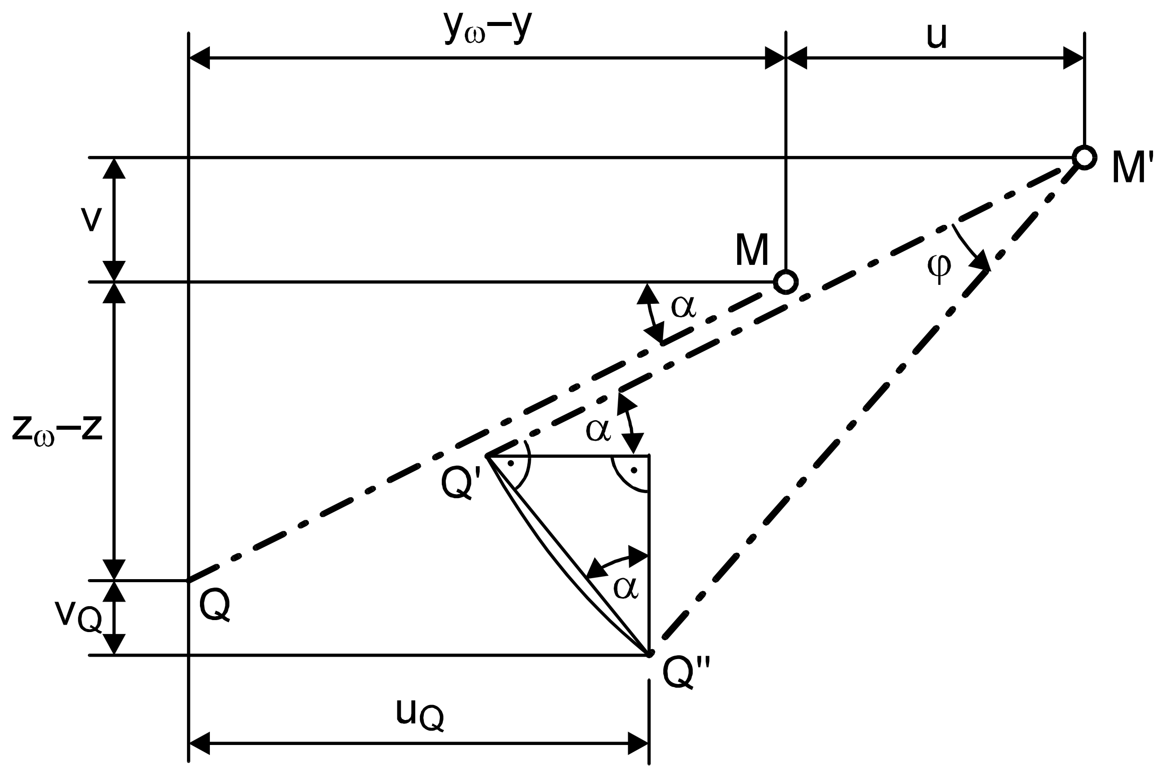
$$v_Q = v - a \cdot \varphi \cdot \cos\alpha$$

$$\sin\alpha = \frac{z_\omega - z}{a}$$

$$\cos\alpha = \frac{y_\omega - y}{a}$$

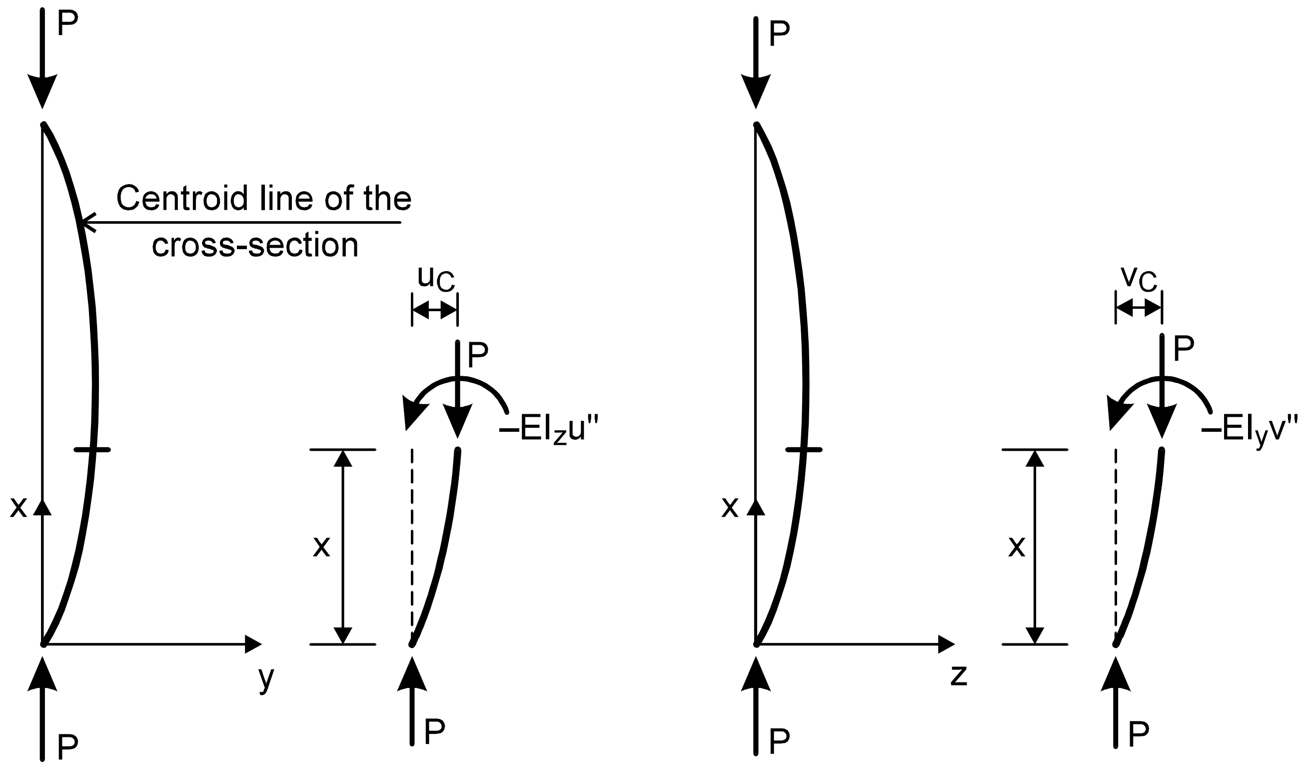
$$u_Q = u + \varphi \cdot (z_\omega - z);$$

$$v_Q = v - \varphi \cdot (y_\omega - y).$$




Centroid C (y=z=0): $u_C = u + \varphi \cdot z_\omega \quad v_C = v - \varphi \cdot y_\omega$

## (b) Equilibrium equations




$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d} x^2} = M_z^k$$

$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d} x^2} = -M_y^k$$

$$EI_\omega \cdot \frac{\mathrm{d}^3 \varphi}{\mathrm{d} x^3} - GI_T \cdot \frac{\mathrm{d} \varphi}{\mathrm{d} x} = -M_T^k$$

Bending moments:

$$M_z^k = P \cdot u_C = P \cdot (u + \varphi \cdot z_\omega)$$

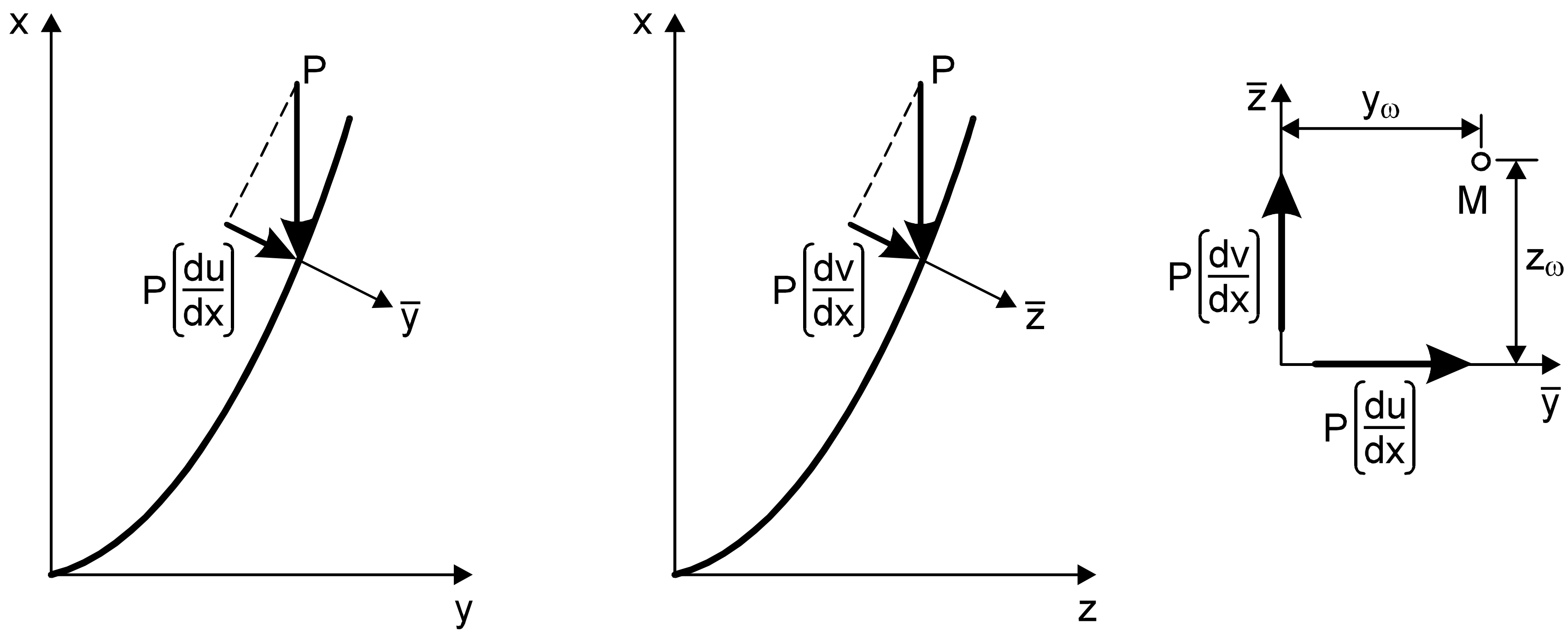
$$M_y^k = P \cdot v_C = P \cdot (v - \varphi \cdot y_\omega)$$

$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d} x^2} + P \cdot (u + \varphi \cdot z_\omega) = 0$$

$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d} x^2} + P \cdot (v - \varphi \cdot y_\omega) = 0$$

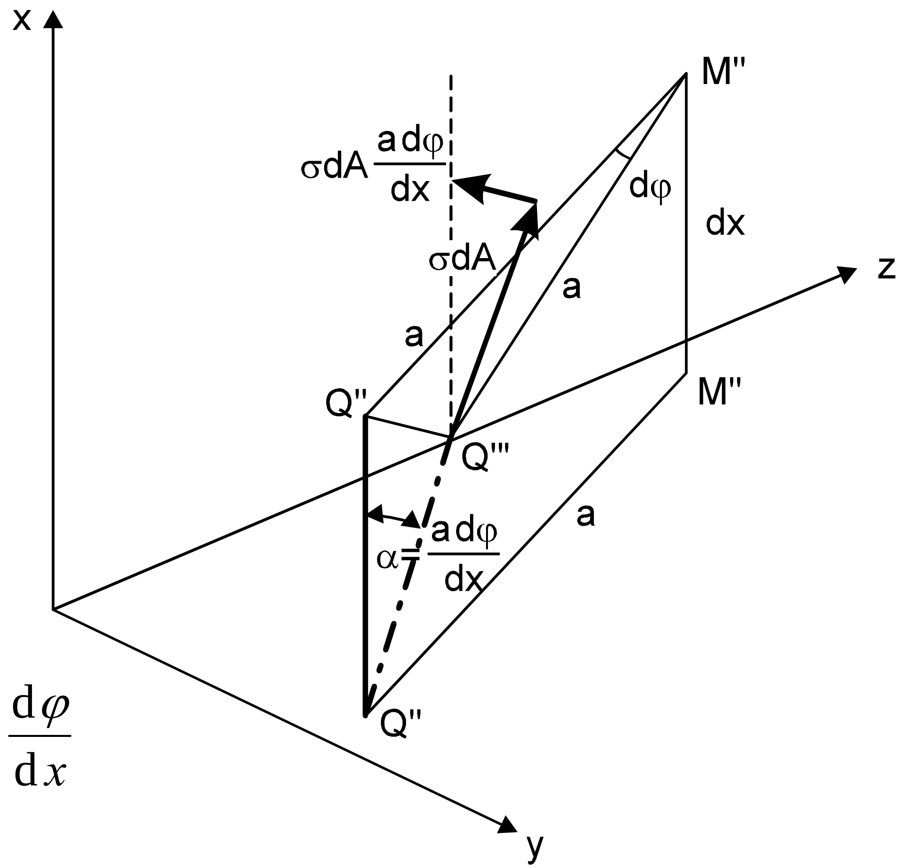
There are two factors which contribute further torque components.

The first component is due to the fact that *P* retains its original direction. In the y-x plane, therefore, *P* has a component *P*du/dx* which acts through the centroid. Together with component of *P* in the z-x plane the column has thus a twisting moment about the shear centre.




$$M_{T,1}^{k} = P \cdot \left( \frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot z_{\omega} - \frac{\mathrm{d}\,v}{\mathrm{d}\,x} \cdot y_{\omega} \right)$$

The second component contribution to $M_T$ is caused by the fact that two cross-sections *dz* apart will warp with respect to each other, and therefore the stress element $\sigma dA$ is inclined by the angle $\alpha$ to the axis *x*.




$$\alpha = a \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

The component of the stress element is

$$(\sigma \cdot \mathrm{d}A) \cdot a \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

And it causes a twist about the shear centre.

$$\mathrm{d}M_{T,2}^{k} = a \cdot (\sigma \cdot \mathrm{d}A) \cdot \left( a \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} \right)$$

$$M_{T,2}^{k} = \frac{\mathrm{d}\varphi}{\mathrm{d}x} \cdot \sigma \cdot \int_A a^2 \,\mathrm{d}A = \frac{\mathrm{d}\varphi}{\mathrm{d}x} \cdot \frac{P}{A} \cdot I_{p\omega} = P \cdot i_{\omega}^2 \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

$$I_{p\omega} = \int_A a^2 \,\mathrm{d}A$$ Moment of polar inertia for the shear centre

$$i_{\omega} = \sqrt{\frac{I_{p\omega}}{A}} \qquad i_{\omega} = \sqrt{i_y^2 + i_z^2 + z_{\omega}^2 + y_{\omega}^2}$$ Radius of gyration for the shear centre

Saint-Venant torsion:

$$M_{SV} = GI_T \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

Non-uniform torsion:

$$M_\omega = -EI_\omega \cdot \frac{\mathrm{d}^3\varphi}{\mathrm{d}x^3}$$

Equilibrium:

$$M_{T,1}^k + M_{T,2}^k = M_{SV} + M_\omega$$

$$P \cdot z_\omega \cdot \frac{\mathrm{d}u}{\mathrm{d}x} - P \cdot y_\omega \cdot \frac{\mathrm{d}v}{\mathrm{d}x} + P \cdot i_\omega^2 \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} + $$
$$+ EI_\omega \cdot \frac{\mathrm{d}^3\varphi}{\mathrm{d}x^3} - GI_T \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} = 0$$

Boundary conditions:

$$u = v = \varphi = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = 0$$

## *(c) Solution of equilibrium equations*

$$EI_z \cdot u'' + P \cdot (u + \varphi \cdot z_\omega) = 0$$

$$EI_y \cdot v'' + P \cdot (v - \varphi \cdot y_\omega) = 0$$

$$P \cdot z_\omega \cdot u' - P \cdot y_\omega \cdot v' + P \cdot i_\omega^2 \cdot \varphi' +$$
$$+ EI_\omega \cdot \varphi''' - G \cdot I_T \cdot \varphi' = 0$$

$$u = C_1 \cdot \sin\frac{\pi}{L}x$$

$$v = C_2 \cdot \sin\frac{\pi}{L}x$$

$$\varphi = C_3 \cdot \sin\frac{\pi}{L}x$$

Substitution of the deflections and their derivatives into the DE gives three homogeneous simultaneous equations. The vanishing of the determinant formed by the coefficients $C_1$, $C_2$ and $C_3$ gives the following buckling conditions:

$$\left[C_1 \cdot \left(P - \frac{\pi^2}{L^2} \cdot EI_z\right) + C_2 \cdot 0 + C_3 \cdot P \cdot z_\omega\right] \cdot \sin\frac{\pi}{L}x = 0;$$

$$\left[C_1 \cdot 0 + C_2 \cdot \left(P - \frac{\pi^2}{L^2} \cdot EI_y\right) + C_3 \cdot (-P \cdot y_\omega)\right] \cdot \sin\frac{\pi}{L}x = 0;$$

$$\left[C_1 \cdot \frac{\pi}{L} \cdot P \cdot z_\omega + C_2 \cdot \frac{\pi}{L} \cdot P \cdot y_\omega + C_3 \cdot \left(\frac{\pi}{L} \cdot P \cdot i_\omega^2 - \frac{\pi^3}{L^3} \cdot EI_\omega - \frac{\pi}{L} \cdot GI_T\right)\right] \cdot \cos\frac{\pi}{L}x = 0.$$

Non-trivial solution:

$$\det\begin{bmatrix} P - \dfrac{\pi^2 \cdot EI_z}{L^2} & 0 & P \cdot z_\omega \\ 0 & P - \dfrac{\pi^2 \cdot EI_y}{L^2} & -P \cdot y_\omega \\ P \cdot z_\omega & -P \cdot y_\omega & i_\omega^2 \cdot \left[P - \dfrac{1}{i_\omega^2} \cdot \left(\dfrac{\pi^2 \cdot EI_\omega}{L^2} + GI_T\right)\right] \end{bmatrix} = 0$$

$$P_{Ez} = \frac{\pi^2 \cdot EI_z}{L^2} = P_z$$

$$P_{Ey} = \frac{\pi^2 \cdot EI_y}{L^2} = P_y$$

$$P_\omega = P_{Ex} = \frac{1}{i_\omega^2} \cdot \left( \frac{\pi^2 \cdot EI_\omega}{L^2} + GI_T \right)$$

Third-order algebraic equation:

$$A_3 \cdot P^3 + A_2 \cdot P^2 + A_1 \cdot P + A_0 = 0$$

$$A_3 = -i_\omega^2 + y_\omega^2 + z_\omega^2;$$

$$A_2 = (P_\omega + P_y + P_z) \cdot i_\omega^2 - z_\omega^2 \cdot P_y - y_\omega^2 \cdot P_z;$$

$$A_1 = -(P_y \cdot P_z + P_z \cdot P_\omega + P_y \cdot P_\omega) \cdot i_\omega^2;$$

$$A_0 = P_y \cdot P_z \cdot P_\omega \cdot i_\omega^2.$$

$$P = q - \frac{1}{3} \cdot \frac{A_2}{A_3}$$

$$q^3 + 3B_1 \cdot q + 2B_0 = 0$$

$$B_1 = \frac{1}{3} \cdot \frac{A_1}{A_3} - \frac{1}{9} \cdot \left( \frac{A_2}{A_3} \right)^2;$$

$$B_0 = \frac{1}{27} \cdot \left( \frac{A_2}{A_3} \right)^2 - \frac{1}{6} \cdot \frac{A_1 \cdot A_2}{A_3^2} + \frac{A_0}{2A_3}.$$

$$q_1 = 2\sqrt{|B_1|} \cdot \cos\kappa;$$

$$q_2 = 2\sqrt{|B_1|} \cdot \cos\left( \frac{2\pi}{3} + \kappa \right); \qquad \cos 3\kappa = -\frac{B_0}{|B_1|^{3/2}}$$

$$q_3 = 2\sqrt{|B_1|} \cdot \cos\left( \frac{2\pi}{3} - \kappa \right),$$

$$P_1 = q_1 - \frac{1}{3} \cdot \frac{A_2}{A_3};$$

$$P_2 = q_2 - \frac{1}{3} \cdot \frac{A_2}{A_3};$$

$$P_3 = q_3 - \frac{1}{3} \cdot \frac{A_2}{A_3}.$$

The critical value of P is always less than either $P_{ez}$, $P_{ey}$ and $P_\omega$, and must therefore be computed!

## *(d) Plane or spatial buckling load is minimum?*

We can prove this as follows.

$$\det\begin{bmatrix} P-P_{Ez} & 0 & P\cdot z_{\omega} \\ 0 & P-P_{Ey} & -P\cdot y_{\omega} \\ P\cdot z_{\omega} & -P\cdot y_{\omega} & i_{\omega}^2\cdot(P-P_{\omega}) \end{bmatrix}=0$$

$$(P-P_{Ez})\cdot(P-P_{Ey})\cdot(P-P_{\omega})-$$

$$-P^2\cdot(P-P_{Ez})\cdot\frac{y_{\omega}^2}{i_{\omega}^2}-$$

$$-P^2\cdot(P-P_{Ey})\cdot\frac{z_{\omega}^2}{i_{\omega}^2}=0$$

Let us call the left side of det. F(P) and assume arbitrarily that:.

$$P_{Ey}<P_{Ez}<P_{\omega}$$

Then:

*(a)* $P=0$

$$\mathrm{F}(P)=-P_{Ey}\cdot P_{Ez}\cdot P_{\omega}<0$$

*(b)* $P=P_{Ey}$

$$\mathrm{F}(P)=-P_{Ey}^2\cdot(P_{Ey}-P_{Ez})\cdot\frac{y_{\omega}^2}{i_{\omega}^2}>0$$

*(c)* $P=P_{Ez}$

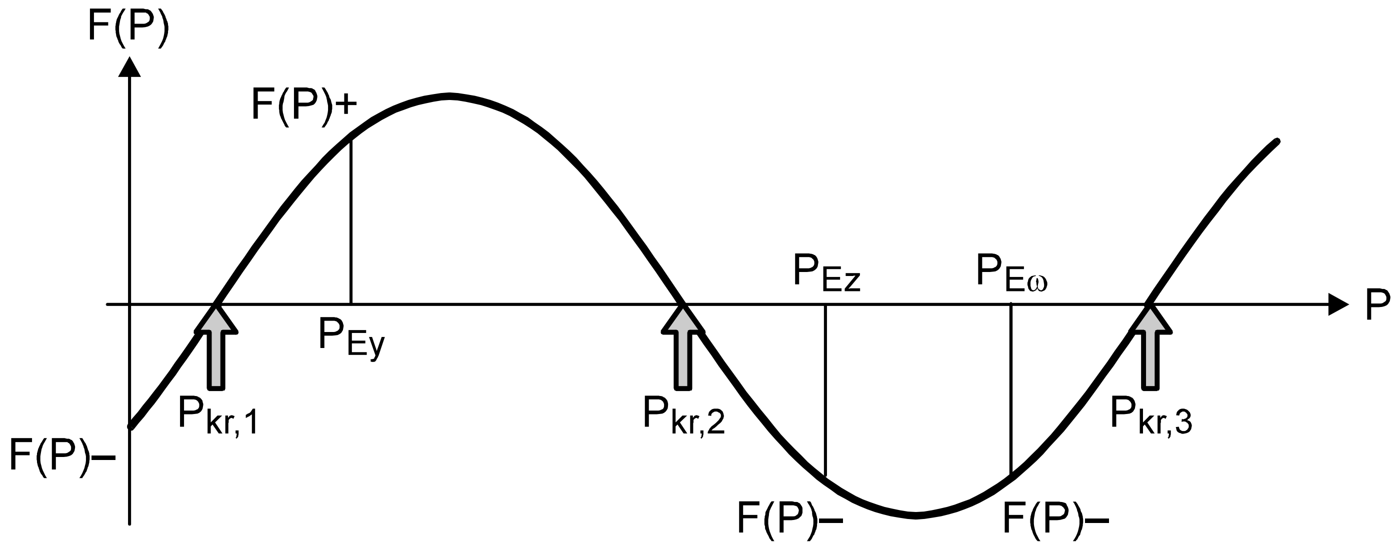
$$\mathrm{F}(P)=-P_{Ez}^2\cdot(P_{Ez}-P_{Ey})\cdot\frac{z_{\omega}^2}{i_{\omega}^2}<0$$

*(d)* $P=P_{\omega}$

$$\mathrm{F}(P)=-P_{\omega}^2\cdot\left[(P_{\omega}-P_{Ez})\cdot\frac{y_{\omega}^2}{i_{\omega}^2}+(P_{\omega}-P_{Ey})\cdot\frac{z_{\omega}^2}{i_{\omega}^2}\right]<0$$

*(e)* $P\to\infty$

$$\mathrm{F}(P)>0$$




$$P_{kr,1} < P_{Ey} < P_{Ez} < P_{\omega}$$

**Critical loads belonging to in-plane and spatial buckling**

## 5.1.2 Energy Method for General Open Cross-section Column

*(a) Strain energy of torsion* [Galambos, 1968] [Chajes, 1974]

St. Venant torsion:

$$M_{SV} = GI_T \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}$$

$$\mathrm{d}L_{b,T} = \frac{1}{2} \cdot M_{SV} \cdot \mathrm{d}\varphi$$

$$\mathrm{d}\varphi = \frac{M_{SV}}{GI_T} \cdot \mathrm{d}x$$

$$\mathrm{d}L_{b,T} = \frac{1}{2} \cdot \frac{M_{SV}^2}{GI_T} \cdot \mathrm{d}x$$

$$\mathrm{d}L_{b,T} = \frac{1}{2} \cdot GI_T \cdot \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 \cdot \mathrm{d}x$$

$$L_{b,T} = \frac{1}{2} \cdot \int_0^L GI_T \cdot \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

Warping torsion:

$$\sigma_\omega = E \cdot \varepsilon_\omega = E \cdot \omega \cdot \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2}$$

$$dL_{b,\omega} = \frac{1}{2} \cdot \int_A \sigma_\omega \cdot \varepsilon_\omega \, \mathrm{d}A$$

$$\mathrm{d}L_{b,\omega} = \frac{1}{2} \cdot E \cdot \left(\frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2}\right)^2 \cdot \int_A \omega^2 \, \mathrm{d}A = \frac{1}{2} \cdot EI_\omega \cdot \left(\frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2}\right)^2$$

$$L_{b,\omega} = \frac{1}{2} \cdot \int_0^L EI_\omega \cdot \left(\frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

## *(b) Total strain energy*

$$L_b = \frac{1}{2}\cdot\int_0^L EI_z \cdot (u'')^2 \,\mathrm{d}x + \frac{1}{2}\cdot\int_0^L EI_y \cdot (v'')^2 \,\mathrm{d}x +$$

$$+\frac{1}{2}\cdot\int_0^L GI_T (\varphi')^2 \,\mathrm{d}x + \frac{1}{2}\cdot\int_0^L EI_\omega (\varphi'')^2 \,\mathrm{d}x.$$

## *(c) The potential energy of external loads*

$$L_k = -\int_A \Delta\sigma \,\mathrm{d}A$$

From the Pythagorean theorem the length *ds* of the deformed element is:

$$\mathrm{d}s = \sqrt{\mathrm{d}u_Q^2 + \mathrm{d}v_Q^2 + \mathrm{d}x^2} =$$

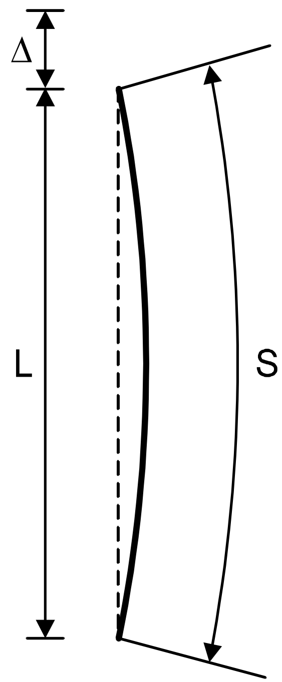
$$= \mathrm{d}x \cdot \sqrt{\left(\frac{\mathrm{d}u_Q}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}v_Q}{\mathrm{d}x}\right)^2 + 1}$$

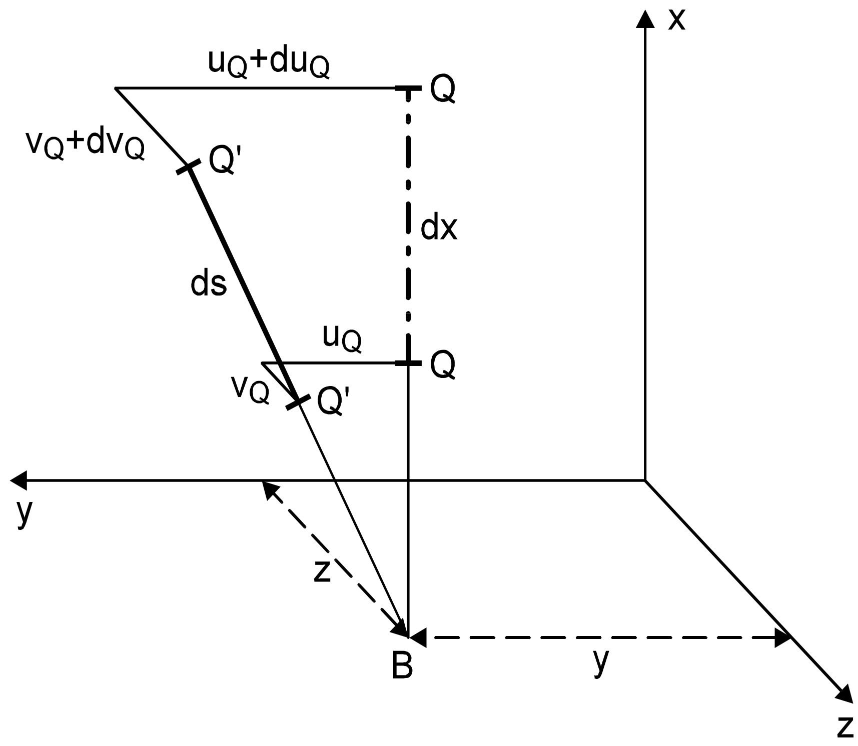
$$\eta \approx 0 \qquad \sqrt{\eta + 1} \approx 1 + \frac{1}{2}\cdot\eta$$

$$\mathrm{d}s = \mathrm{d}x \cdot \left[\frac{1}{2}\cdot\left(\frac{\mathrm{d}u_Q}{\mathrm{d}x}\right)^2 + \frac{1}{2}\cdot\left(\frac{\mathrm{d}v_Q}{\mathrm{d}x}\right)^2 + 1\right]$$

$$S = \int_0^L \mathrm{d}s = \int_0^L \left[\frac{1}{2}\cdot\left(\frac{\mathrm{d}u_Q}{\mathrm{d}x}\right)^2 + \frac{1}{2}\cdot\left(\frac{\mathrm{d}v_Q}{\mathrm{d}x}\right)^2 + 1\right]\mathrm{d}x$$

$$\Delta = S - L = \frac{1}{2}\cdot\int_0^L \left[\left(\frac{\mathrm{d}u_Q}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}v_Q}{\mathrm{d}x}\right)^2\right]\mathrm{d}x$$



The total displacements of the fibre at *(y,z)* are therefore:

$$u_Q = u + \varphi \cdot (z_\omega - z) \qquad v_Q = v - \varphi \cdot (y_\omega - y)$$

$$\Delta = \frac{1}{2} \cdot \int_0^L \Big[ (u')^2 + 2u' \cdot \varphi' \cdot (z_\omega - z) + (\varphi')^2 \cdot (z_\omega^2 - 2z_\omega \cdot z + z^2) +$$

$$+ (v')^2 - 2v' \cdot \varphi' \cdot (y_\omega - y) + (\varphi')^2 \cdot (y_\omega^2 - 2y_\omega \cdot y + y^2) \Big] \mathrm{d}\, x.$$

To simplify this expression, we make use of the following relations:

$$\sigma = \frac{P}{A} \qquad \int_{(A)} \mathrm{d}\, A = A \quad \int_{(A)} z \, \mathrm{d}\, A = 0 \quad \int_{(A)} y \, \mathrm{d}\, A = 0$$

$$\int_{(A)} y^2 \, \mathrm{d}\, A = I_z \qquad \int_{(A)} z^2 \, \mathrm{d}\, A = I_y \qquad i_\omega = i_y^2 + i_z^2 + y_\omega^2 + z_\omega^2$$

$$L_k = -\frac{1}{2} \cdot \int_0^L \int_{(A)} \sigma \cdot \Big[ (u')^2 + 2u' \cdot \varphi' \cdot z_\omega + (\varphi')^2 \cdot (z_\omega^2 + z^2) +$$

$$+ (v')^2 - 2v' \cdot \varphi' \cdot y_\omega + (\varphi')^2 \cdot (y_\omega^2 + y^2) \Big] \mathrm{d}\, A \, \mathrm{d}\, x.$$

The potential energy of external loads:

$$L_k = -\frac{P}{2} \cdot \int_0^L \Big[ (u')^2 + (v')^2 + i_\omega^2 \cdot (\varphi')^2 + 2z_\omega \cdot u' \cdot \varphi' - 2y_\omega \cdot v' \cdot \varphi' \Big] \mathrm{d}\, x$$

## (d) Simple supported column

Boundary conditions: $x=0$, $x=L$ $\longrightarrow$ $u=v=0$, $\dfrac{d^2 u}{d x^2}=\dfrac{d^2 v}{d x^2}=0$, $\varphi=\dfrac{d^2\varphi}{d x^2}=0$

Assumed buckling shape:

$$u=C_1\cdot\sin\frac{\pi x}{L}\qquad v=C_2\cdot\sin\frac{\pi x}{L}\qquad \varphi=C_3\cdot\sin\frac{\pi x}{L}$$

Strain energy:

$$L_b=\frac{1}{2}\cdot\int_0^L EI_z\cdot C_1^2\cdot\frac{\pi^4}{L^4}\cdot\sin^2\frac{\pi\cdot x}{L}\,d x+\frac{1}{2}\cdot\int_0^L EI_y\cdot C_2^2\cdot\frac{\pi^4}{L^4}\cdot\sin^2\frac{\pi\cdot x}{L}\,d x+$$

$$+\frac{1}{2}\cdot\int_0^L GI_T\cdot C_3\cdot\frac{\pi^2}{L^2}\cdot\cos^2\frac{\pi\cdot x}{L}\,d z+\frac{1}{2}\cdot\int_0^L EI_\omega\cdot C_3\cdot\frac{\pi^4}{L^4}\cdot\sin^2\frac{\pi\cdot x}{L}\,d x.$$

$$\int_0^L\cos^2\frac{\pi\cdot x}{L}\,d z=\int_0^L\sin^2\frac{\pi\cdot x}{L}\,d x=\frac{L}{2}$$

$$L_b=\frac{1}{4}\cdot\frac{\pi^2}{L}\cdot\left[C_1^2\cdot\frac{EI_z\cdot\pi^2}{L^2}+C_2^2\cdot\frac{EI_y\cdot\pi^2}{L^2}+C_3^2\cdot\left(GI_T+\frac{EI_\omega\cdot\pi^2}{L^2}\right)\right]$$

$$P_{Ey} = \frac{\pi^2 \cdot EI_y}{L^2} \qquad P_{Ez} = \frac{\pi^2 \cdot EI_z}{L^2} \qquad P_\omega = \frac{1}{i_\omega^2} \cdot \left( GI_T + \frac{\pi^2 \cdot EI_\omega}{L^2} \right)$$

$$L_b = \frac{1}{4} \cdot \frac{\pi^2}{L} \cdot (C_1^2 \cdot P_{Ez} + C_2^2 \cdot P_{Ey} + C_3^2 \cdot P_\omega)$$

Potential energy of the external load:

$$L_k = -\frac{P \cdot \pi^2}{4L} (C_1^2 + C_2^2 + C_3^2 \cdot i_\omega^2 + 2C_1 \cdot C_3 \cdot z_\omega - 2C_2 \cdot C_3 \cdot y_\omega)$$

Total potential energy:

$$L_b + L_k = \delta^2 \Pi = \frac{\pi^2}{4L} \cdot \left[ C_1^2 \cdot (P_{Ez} - P) + C_2^2 \cdot (P_{Ey} - P) + \right.$$
$$\left. + C_3^2 \cdot i_\omega^2 \cdot (P_\omega - P) + 2C_1 \cdot C_3 \cdot P \cdot z_\omega - 2C_2 \cdot C_3 \cdot P \cdot y_\omega \right].$$

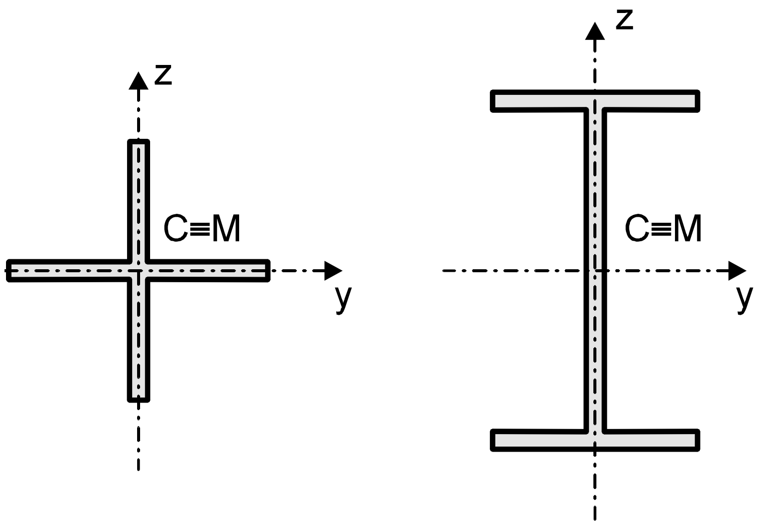
Stationary value $\rightarrow$ all the derivatives vanish:

$$\frac{\partial(\delta^2 \Pi)}{\partial C_1} = C_1 \cdot (P_{Ez} - P) + C_3 \cdot P \cdot z_\omega = 0;$$

$$\frac{\partial(\delta^2 \Pi)}{\partial C_2} = C_2 \cdot (P_{Ey} - P) + C_3 \cdot P \cdot y_\omega = 0; \qquad \longrightarrow \qquad \det \begin{bmatrix} P_{Ey} - P & 0 & P \cdot z_\omega \\ 0 & P_{Ex} - P & -P \cdot y_\omega \\ P \cdot z_\omega & -P \cdot y_\omega & i_\omega^2 \cdot (P_\omega - P) \end{bmatrix} = 0$$

$$\frac{\partial(\delta^2 \Pi)}{\partial C_3} = C_1 \cdot (P \cdot z_\omega) - C_2 \cdot (P \cdot y_\omega) + C_3 \cdot i_\omega^2 \cdot (P_\omega - P) = 0.$$

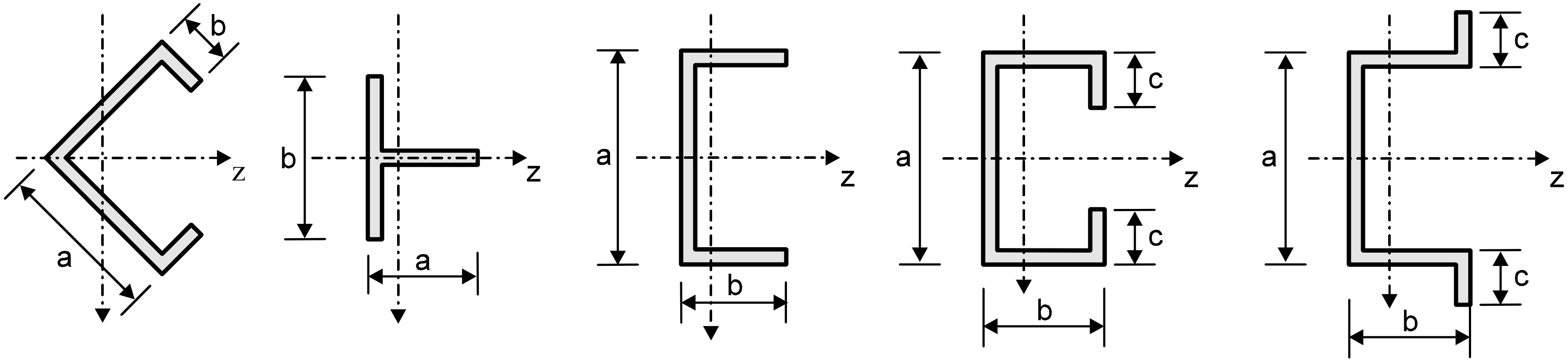
## 5.1.3 Double-Symmetric Open Cross-section




$$y_\omega = z_\omega = 0$$

$$P_{kr,1} = P_{Ez} \qquad P_{kr,2} = P_{Ey} \qquad P_{kr,3} = P_\omega$$

## 5.1.4 Mono-Symmetric Open Cross-section




$$y_\omega = 0$$

Two possibilities considered:

*(a) Euler-type flexural in-plane buckling in x-z plane:*

$$P_{cr,1} = P_{Ey}$$

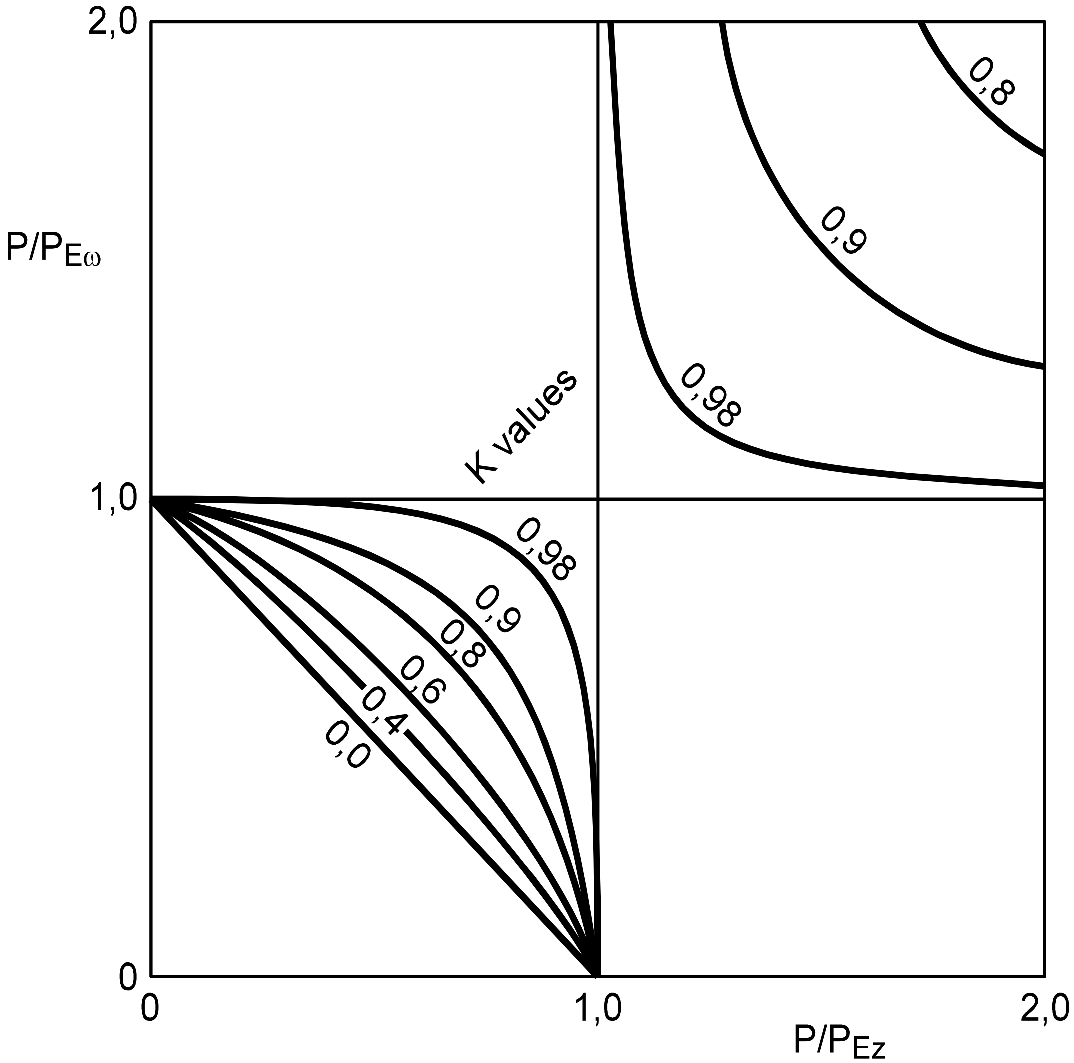
*(b) Flexural-torsional buckling:*

$P_{cr,2}$ and $P_{cr,3}$

$$(P - P_{Ez}) \cdot (P - P_{\omega}) - P^2 \cdot \frac{z_{\omega}^2}{i_{\omega}^2} = 0$$

$$\frac{P}{P_{Ez}} + \frac{P}{P_{\omega}} - K \cdot \frac{P^2}{P_{Ez} P_{\omega}} = 1$$

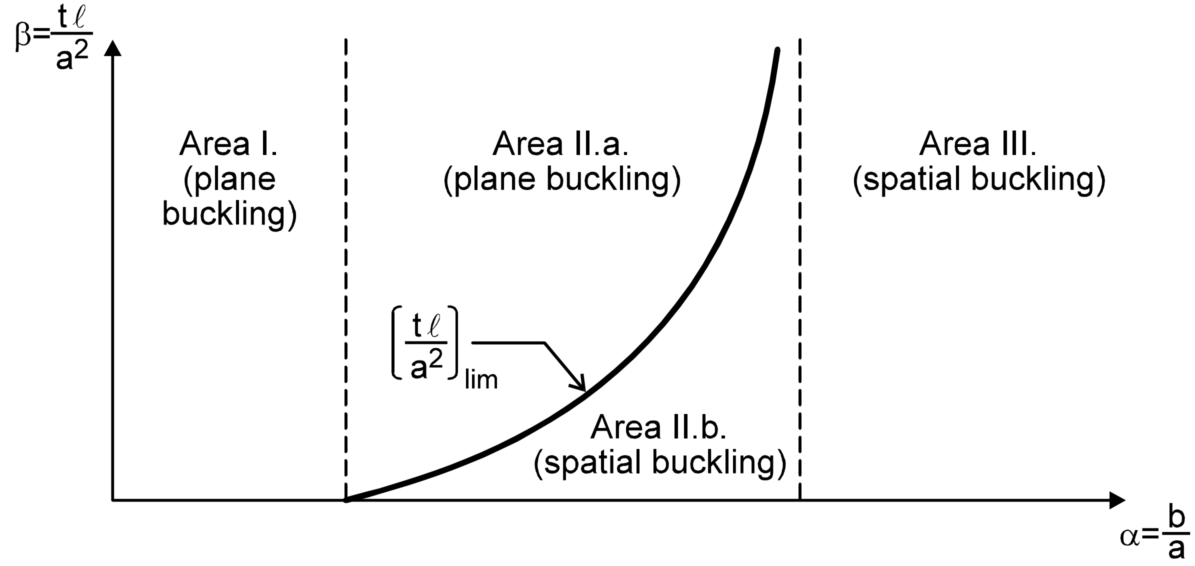
$$K = 1 - \left( \frac{z_{\omega}}{i_{\omega}} \right)^2$$

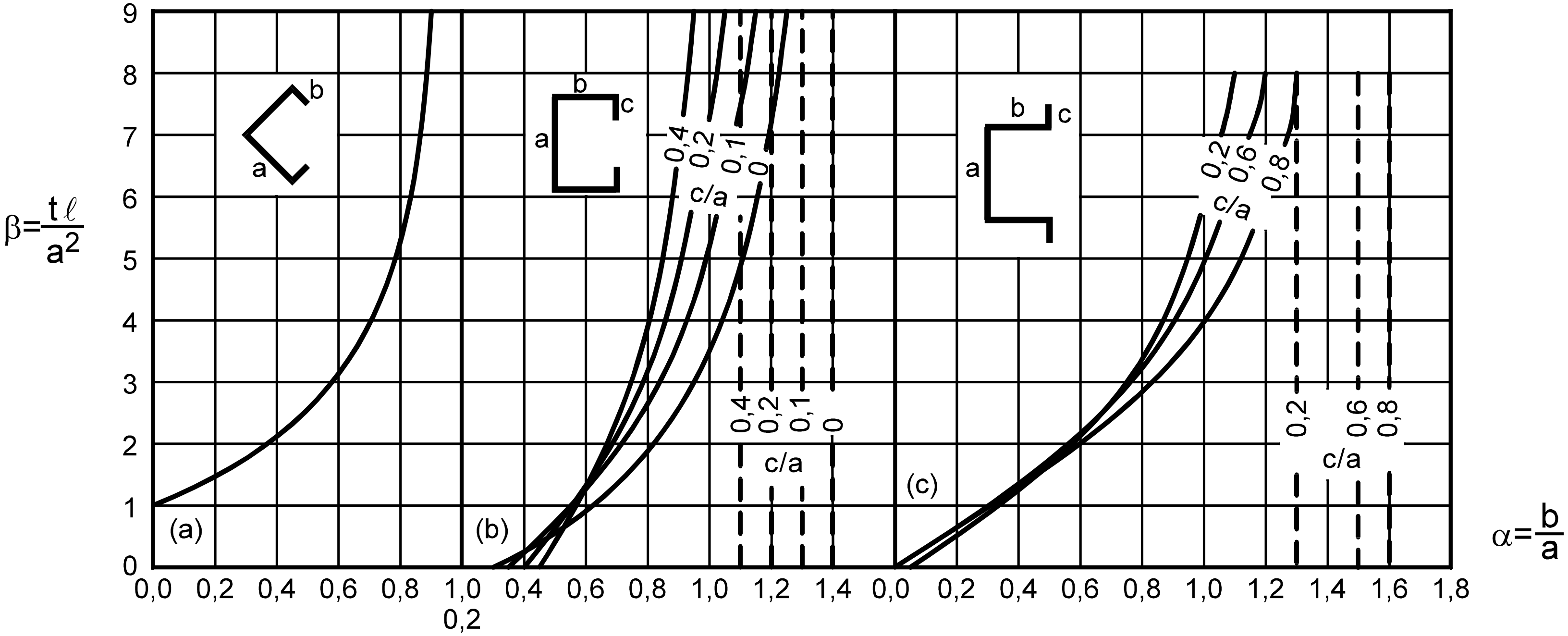



[Gerard, Becker, 1957] [Chajes, Winter, 1965]

Parameters to be used:

$$\alpha = b / a \qquad \beta = t \cdot l / a^2$$




**Limit curves to separate plane and spatial buckling**

## Design process:

$$P_{cr,2} < P_{cr,3} \qquad \rho = \sqrt{\frac{P_{Ez}}{P_{cr,2}}}$$

$$\left(1 - \frac{z_\omega^2}{i_\omega^2}\right) \cdot P^2 - (P_{Ez} + P_\omega) \cdot P + P_{Ez} \cdot P_\omega = 0$$

$$C^2 = \frac{i_\omega^2 \cdot P_\omega}{P_{Ez}} \qquad A = \frac{P_{Ez}}{P_\omega} = \frac{i_\omega^2}{C^2} \qquad B = \frac{z_\omega}{i_\omega}$$

$$(1 - B^2) \cdot P^2 - (P_{Ez} + P_\omega) \cdot P + P_{Ez} \cdot P_\omega = 0$$

$$P_{cr,2,3} = \frac{1}{2 \cdot (1 - B^2)} \cdot \left[P_{Ez} + P_\omega \pm \sqrt{(P_{Ez} + P_\omega)^2 - 4 \cdot (1 - B^2) \cdot P_{Ez} \cdot P_\omega}\right]$$

$$P_{cr,2} = \frac{P_\omega}{2 \cdot (1 - B^2)} \cdot \left[1 + A - \sqrt{(1 + A)^2 - 4 \cdot (1 - B^2) \cdot A}\right]$$

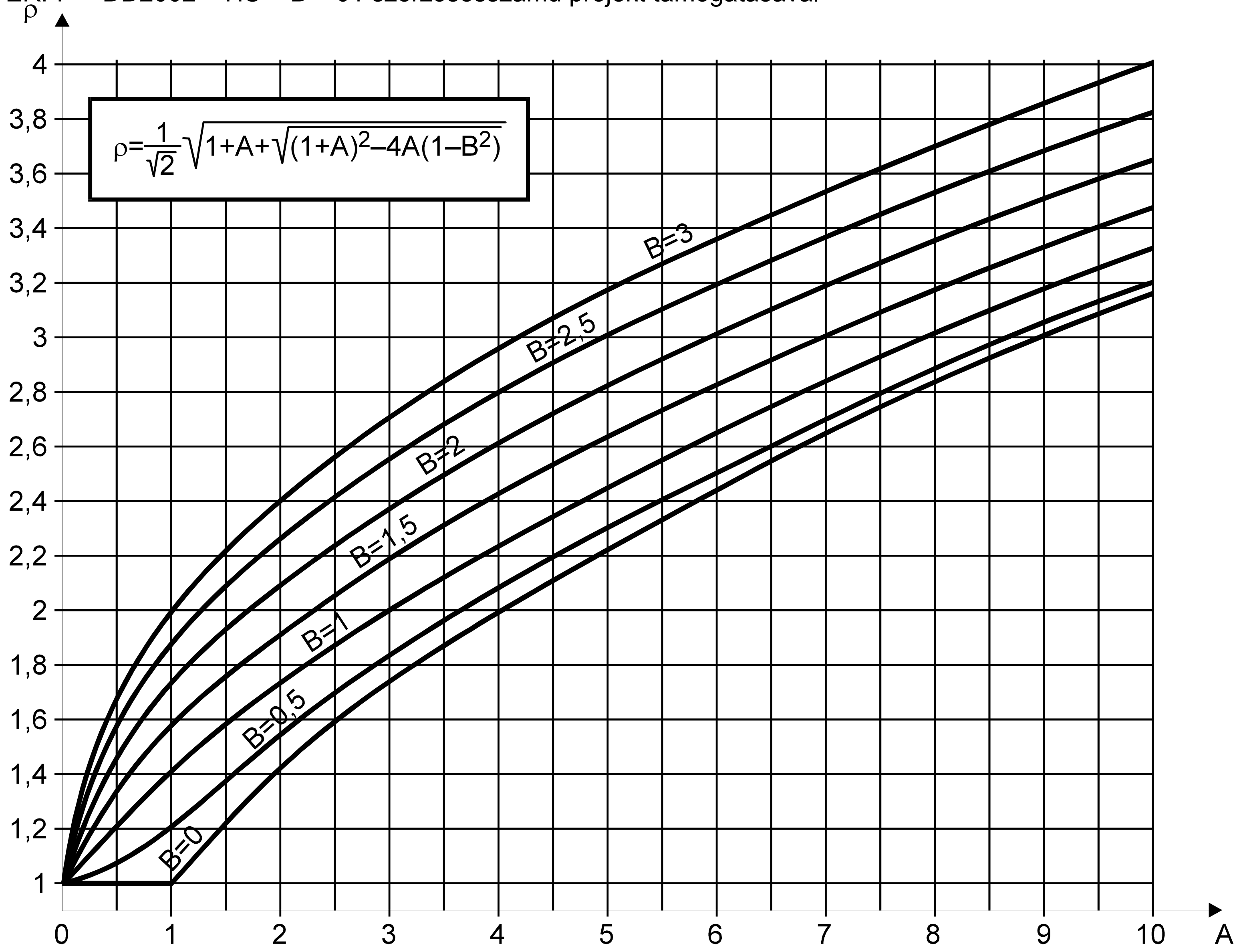
$$\rho = \sqrt{\frac{2P_{Ez} \cdot (1 - B^2)}{P_\omega \cdot \left[1 + A - \sqrt{(1 + A)^2 - 4 \cdot (1 - B^2) \cdot A}\right]}}$$

$$\rho = \frac{1}{\sqrt{2}} \cdot \sqrt{1 + A + \sqrt{(1 + A)^2 - 4 \cdot (1 - B^2) \cdot A}}$$

$$\lambda_z = l_z / i_z \longrightarrow \lambda_i = \rho \cdot \lambda_z$$

Ideal slenderness

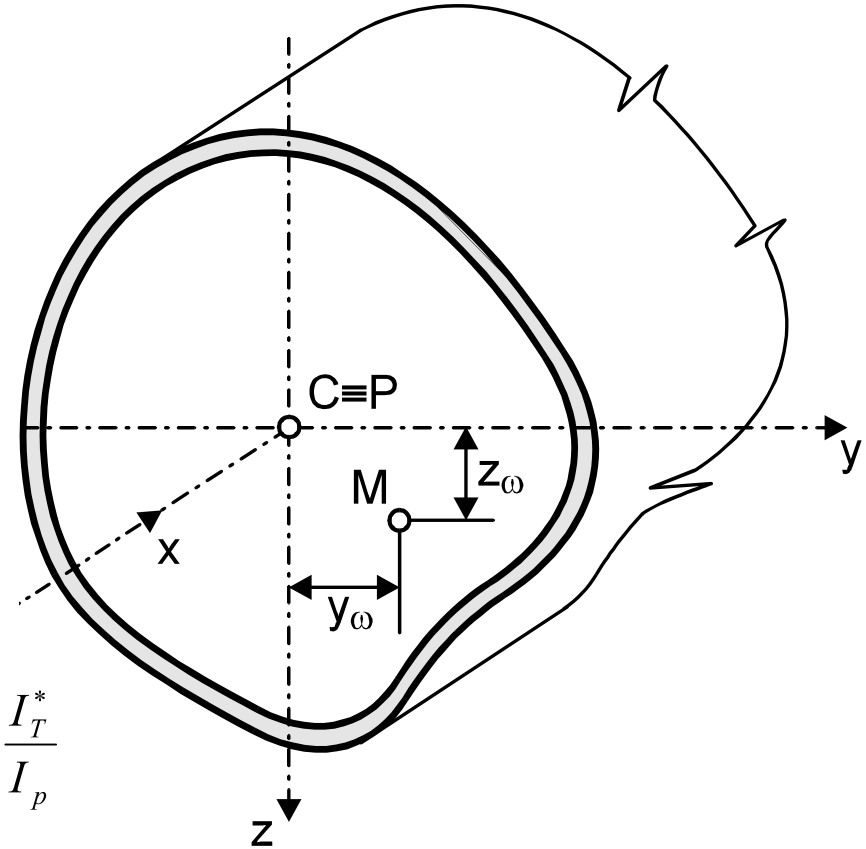
*(Values of ρ coefficient are shown in the next slide)*




**Values of ρ coefficient for spatial (flexural-torsional) buckling**

## 5.1.5 Column with Closed Cross-section

[Hunyadi, 1962]




$$EI_z \cdot \frac{d^2 u}{d x^2} = M_z^k;$$

$$EI_y \cdot \frac{d^2 v}{d x^2} = -M_y^k;$$

$$EI_\omega^* \cdot \frac{d^3 \phi}{d x^3} - GI_T^* \cdot \frac{d \varphi}{d x} = -M_T^k;$$

$$GI_p \cdot \frac{d \varphi}{d x} + G \cdot (I_T^* - I_p) \cdot \frac{d \phi}{d x} = M_T^k.$$

$$\frac{d^4 \phi}{d x^4} = \frac{1}{\mu} \cdot \left( \frac{d^4 \varphi}{d x^4} - \frac{d^3 M_T^k}{d x^3} \cdot \frac{1}{GI_p} \right) \qquad \mu = 1 - \frac{I_T^*}{I_p}$$

$$EI_z \cdot \frac{d^2 u}{d x^2} = M_z^k;$$

$$EI_y \cdot \frac{d^2 v}{d x^2} = -M_y^k;$$

$$EI_\omega^* \cdot \frac{d^4 \varphi}{d x^4} - \mu \cdot GI_T^* \cdot \frac{d^2 \varphi}{d x^2} = -\mu \cdot \frac{d M_T^k}{d x} + \frac{EI_\omega^*}{GI_p} \cdot \frac{d^3 M_T^k}{d x^3}.$$

$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + P \cdot (u + \varphi \cdot z_\omega) = 0;$$

$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + P \cdot (v - \varphi \cdot y_\omega) = 0;$$

$$EI_\omega^* \cdot \frac{\mathrm{d}^4 \varphi}{\mathrm{d}x^4} - \mu \cdot GI_T^* \cdot \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} + \mu \cdot P \cdot \left( \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \cdot z_\omega - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \cdot y_\omega + \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} \cdot i_\omega^2 \right) -$$

$$- \frac{EI_\omega^*}{GI_p} \cdot P \cdot \left( \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \cdot z_\omega - \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} \cdot y_\omega + \frac{\mathrm{d}^4 \varphi}{\mathrm{d}x^4} \cdot i_\omega^2 \right) = 0.$$

$$P_{Ez} = \frac{\pi^2 \cdot EI_z}{L^2} = P_z;$$

$$P_{Ey} = \frac{\pi^2 \cdot EI_y}{L^2} = P_y;$$

$$P_\omega^* = P_{Ex} = \frac{1}{\rho \cdot i_\omega^2} \cdot \left( \frac{\pi^2 \cdot EI_\omega^*}{L^2} + \mu \cdot GI_T^* \right)$$

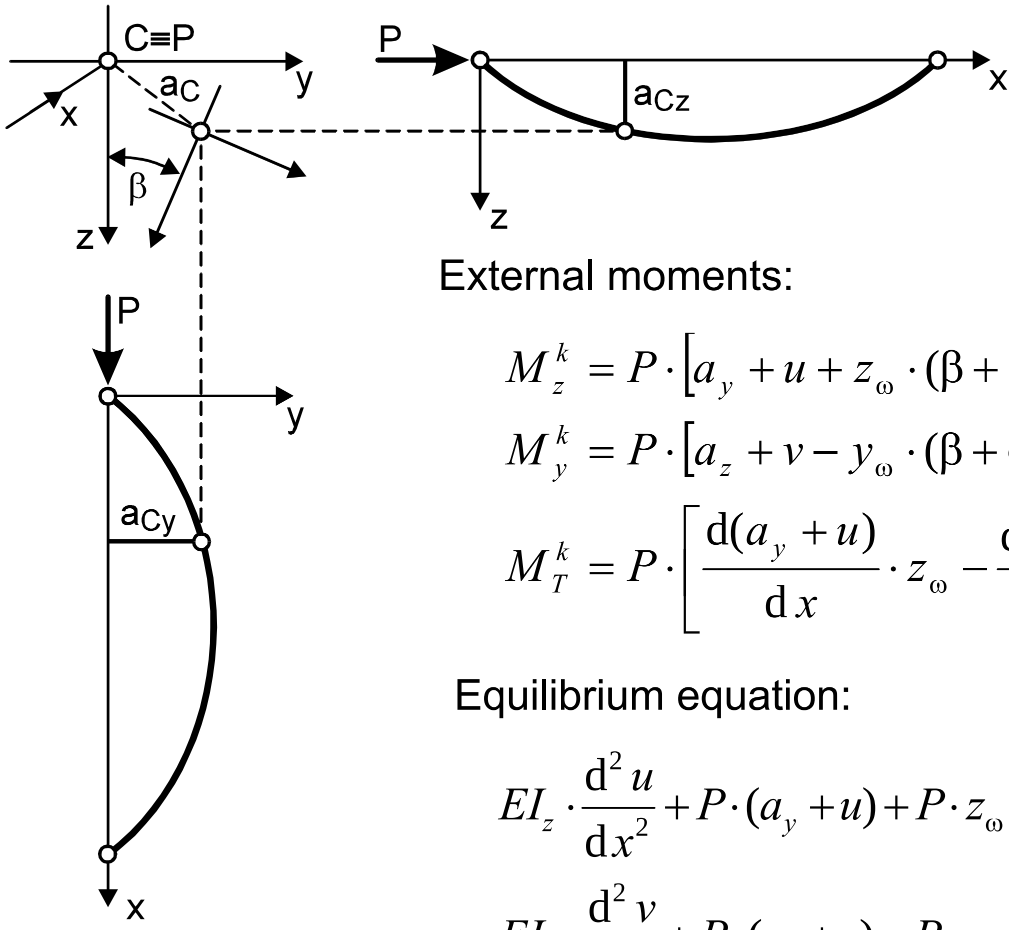
$$\det \begin{bmatrix} P - P_{Ez} & 0 & P \cdot z_\omega \\ 0 & P - P_{Ey} & -P \cdot y_\omega \\ P \cdot z_\omega & -P \cdot y_\omega & i_\omega^2 \cdot (P - P_\omega^*) \end{bmatrix} = 0$$

$$\rho = \mu + \frac{EI_\omega^*}{GI_p} \cdot \frac{\pi^2}{L^2}$$

Készült az ERFP – DD2002 – HU – B – 01 szerzősésszámú projekt támogatásával

# 5.1.6 Lateral-Torsional Buckling of Columns with Imperfections

[Hunyadi, 1962] [Iványi, Hunyadi, 1988]




$$a_y = A_0 \cdot \sin \frac{\pi}{L} x$$

$$a_z = B_0 \cdot \sin \frac{\pi}{L} x$$

$$\beta = C_0 \cdot \sin \frac{\pi}{L} x$$

External moments:

$$M_z^k = P \cdot \left[ a_y + u + z_\omega \cdot (\beta + \varphi) \right];$$

$$M_y^k = P \cdot \left[ a_z + v - y_\omega \cdot (\beta + \varphi) \right];$$

$$M_T^k = P \cdot \left[ \frac{\mathrm{d}(a_y + u)}{\mathrm{d}x} \cdot z_\omega - \frac{\mathrm{d}(a_z + v)}{\mathrm{d}x} \cdot y_\omega + \frac{\mathrm{d}(\beta + \varphi)}{\mathrm{d}x} \cdot i_\omega^2 \right].$$

Equilibrium equation:

$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + P \cdot (a_y + u) + P \cdot z_\omega \cdot (\beta + \varphi) = 0;$$

$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + P \cdot (a_z + v) - P \cdot y_\omega \cdot (\beta + \varphi) = 0;$$

$$EI_\omega \cdot \frac{\mathrm{d}^3 \varphi}{\mathrm{d}x^3} + P \cdot \frac{\mathrm{d}(a_y + u)}{\mathrm{d}x} \cdot z_\omega - P \cdot \frac{\mathrm{d}(a_z + v)}{\mathrm{d}x} \cdot y_\omega + P \cdot i_\omega^2 \cdot \frac{\mathrm{d}(\beta + \varphi)}{\mathrm{d}x} - GI_T \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} = 0.$$

Solution:

$$(P_{Ez} - P) \cdot A - P \cdot z_\omega \cdot C = P \cdot (A_0 + z_\omega \cdot C_0);$$

$$(P_{Ey} - P) \cdot B + P \cdot y_\omega \cdot C = -P \cdot (B_0 - y_\omega \cdot C_0);$$

$$-P \cdot \frac{z_\omega}{i_\omega^2} \cdot A + P \cdot \frac{z_\omega}{i_\omega^2} \cdot B + (P_\omega - P) \cdot C = P \cdot \left( \frac{A_0 \cdot z_\omega}{i_\omega^2} - \frac{B_0 \cdot y_\omega}{i_\omega^2} + C_0 \right).$$

$$\frac{A}{B} = -\frac{z_\omega}{y_\omega} \cdot \frac{P_{Ey} - P_{kr,1}}{P_{Ez} - P_{kr,1}}$$

$$\frac{B}{C} = -\frac{P - P_{kr,1}}{P_{Ez} \cdot z_\omega} \cdot \frac{i_\omega^2}{\frac{z_\omega}{y_\omega} \cdot \frac{P_{Ey} - P_{kr,1}}{P_{Ez} - P_{kr,1}} + \frac{y_\omega}{z_\omega}}$$

$$\frac{A}{C} = \frac{P_\omega - P_{kr,1}}{P_{kr,1} \cdot y_\omega} \cdot \frac{i_\omega^2}{\frac{y_\omega}{z_\omega} \cdot \frac{P_{Ez} - P_{kr,1}}{P_{Ey} - P_{kr,1}} + \frac{z_\omega}{y_\omega}}$$

$$\frac{A_0}{B_0} = \frac{A}{B} \qquad \frac{B_0}{C_0} = \frac{B}{C} \qquad \frac{A_0}{C_0} = \frac{A}{C}$$

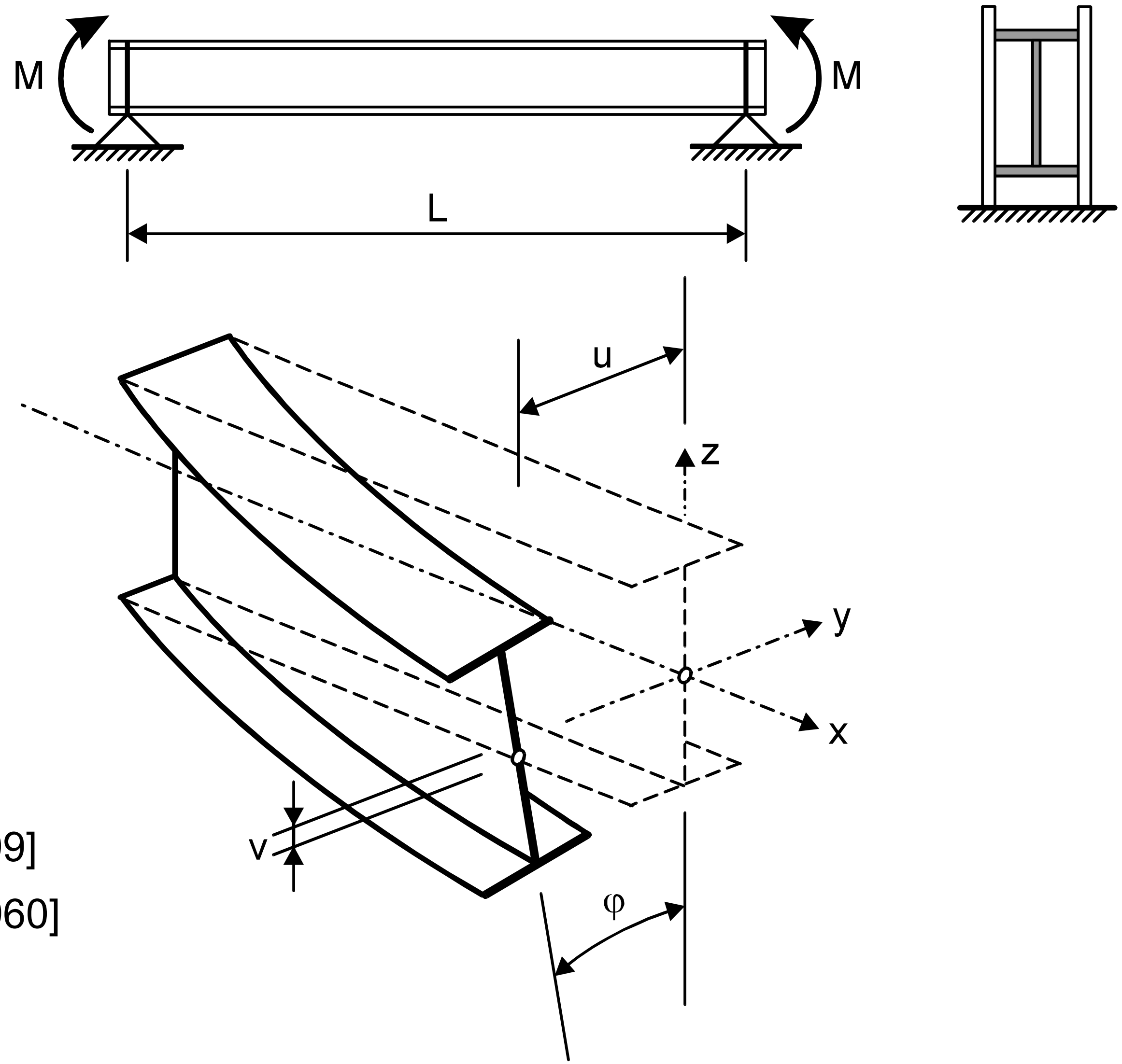
$$A = A_0 \cdot \frac{P}{P_{kr,1} - P}$$

$$B = B_0 \cdot \frac{P}{P_{kr,1} - P} \qquad C = C_0 \cdot \frac{P}{P_{kr,1} - P}$$

Open cross-section: $P_\omega$

Closed cross-section: $P_\omega^*$

# 5.2. Lateral-Torsional Buckling of Beams




[Prandtl, 1899] [Mitchell, 1899]

[Timoshenko, 1910] [Lee, 1960]

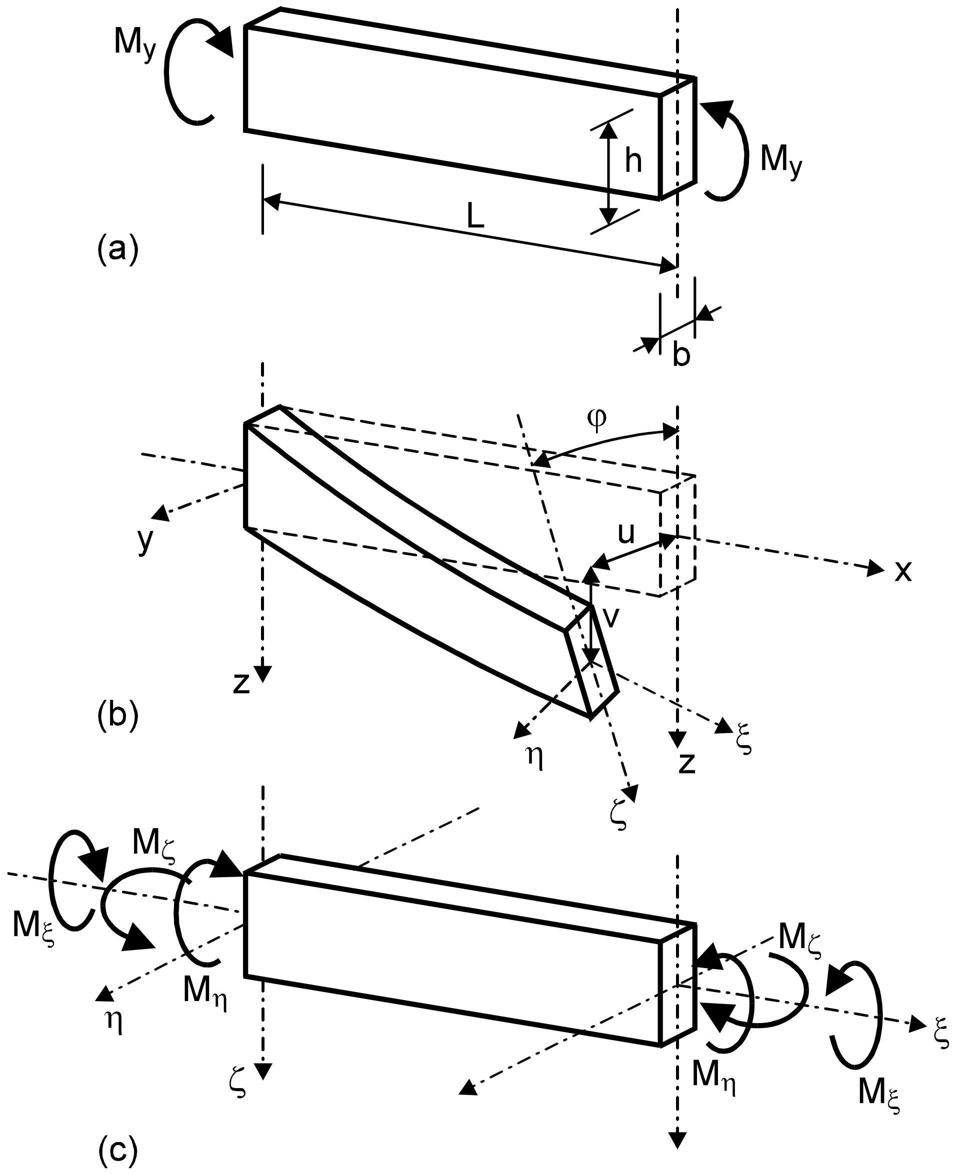
[Nethercot, 1983]

[Trahair, Bradford, 1988]

# 5.2.1 Equilibrium Method for Beams in Pure Bending

## *(a) Rectangular Cross-section*

[Chen, Lui, 1987]




Boundary conditions:

$x = 0$, $x = L$ →

$$u = v = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0$$

$$\varphi = \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} = 0$$

Bending and torsion:

$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -M_\eta$$

$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = M_\zeta$$

$$GI_T \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} = M_\xi$$

For rectangular cross-section it is supposed: $I_\omega \approx 0$

Moment components:

$$M_\eta = M_y \cdot \cos\varphi = M_y$$

$$M_\zeta = M_y \cdot \cos(\varphi + 90°) = -M_y \cdot \sin\varphi = -M_y \cdot \varphi$$

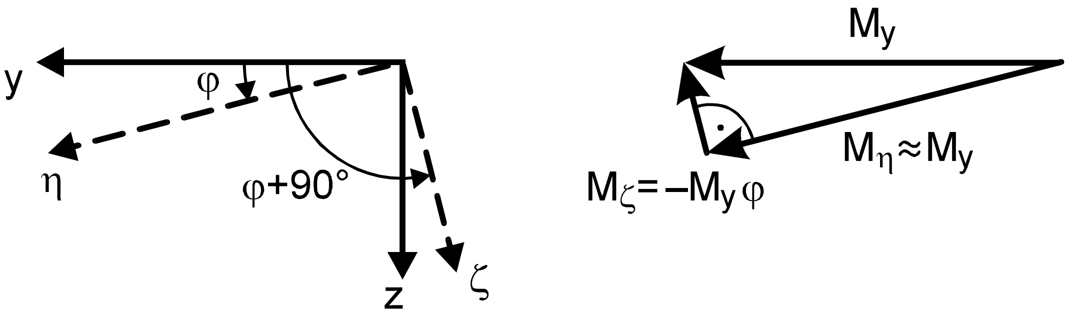
$$M_\xi = M_y \cdot \cos\left(90° - \frac{\mathrm{d}u}{\mathrm{d}x}\right) = M_y \cdot \sin\frac{\mathrm{d}u}{\mathrm{d}x} = M_y \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Substitution of the expressions in DE, leads to the following equations:

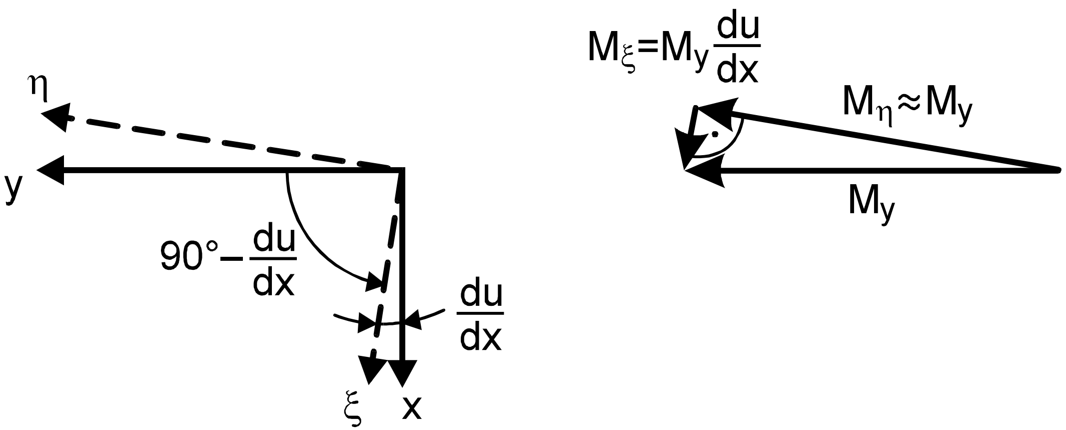
$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + M_y = 0$$

$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + M_y \cdot \varphi = 0$$

$$GI_T \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} - M_y \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = 0$$




(a)




(b)

$$GI_T \cdot \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{M_y^2}{EI_z} \cdot \varphi = 0$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + k^2 \cdot \varphi = 0$$

$$k^2 = \frac{M_y^2}{GI_T \cdot EI_z}$$

$$\varphi = A \cdot \sin kx + B \cdot \cos kx$$

Boundary conditions:

$$\left.\begin{matrix} x = 0 \\ x = L \end{matrix}\right| \rightarrow \varphi = 0$$

$$B = 0 \qquad A \cdot \sin kL = 0$$

Non-trivial solution:

$$\boxed{M_{cr} = \frac{\pi}{L} \cdot \sqrt{GI_T \cdot EI_z}}$$

$$\sigma_{cr} = \frac{M_{kr}}{W_y} = \frac{\pi}{L \cdot W_y} \cdot \sqrt{GI_T \cdot EI_z}$$

For rectangular cross-section:

$$I_T = \frac{h \cdot b^3}{3} \qquad W_y = \frac{2I_y}{h} \qquad I_y = \frac{b \cdot h^3}{12}$$

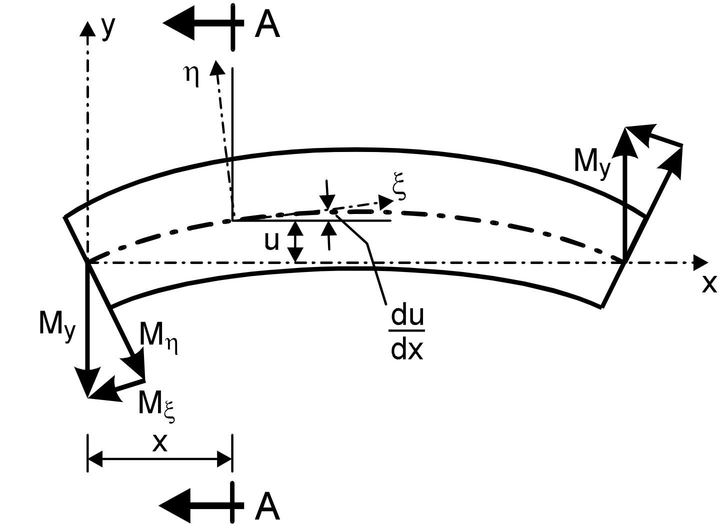
$$\boxed{\begin{aligned} \sigma_{cr} &= \frac{\pi}{L} \cdot \sqrt{G \cdot E} \cdot \sqrt{\frac{h \cdot b^3}{3} \cdot \frac{I_z \cdot h^2}{4I_y} \cdot \frac{12}{b \cdot h^3}} = \\ &= \frac{\pi \cdot \sqrt{G \cdot E}}{\frac{L}{b}} \cdot \sqrt{\frac{I_z}{I_y}} \end{aligned}}$$

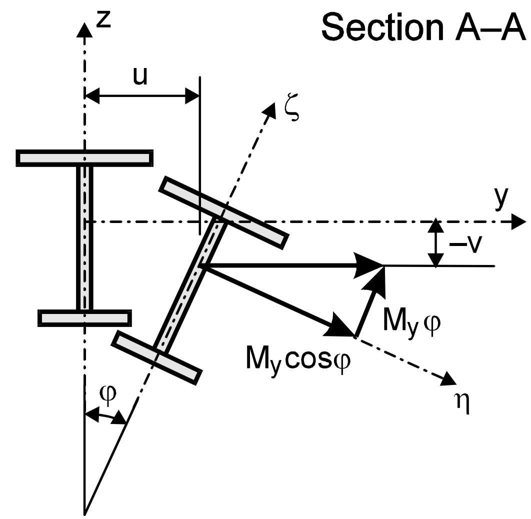
## *(b) "I" Cross-section* [Timoshenko, 1910]

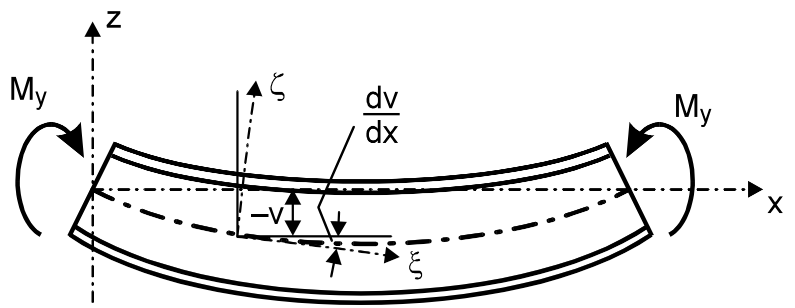
Initial assumptions and boundary conditions are the same as for rectangular section, but: $I_\omega \neq 0$

↓

$$M_\omega = -EI_\omega \cdot \frac{\mathrm{d}^3 \varphi}{\mathrm{d}x^3}$$




$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + M_y \cdot \varphi = 0$$

$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + M_y = 0$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -\frac{M_y}{EI_z} \cdot \varphi$$

$$GI_T \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} - EI_\omega \cdot \frac{\mathrm{d}^3 \varphi}{\mathrm{d}x^3} - M_y \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

$$EI_\omega \cdot \frac{\mathrm{d}^4 \varphi}{\mathrm{d}x^4} - GI_T \cdot \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} - \frac{M_y^2}{EI_z} \cdot \varphi = 0$$

$$2\alpha = \frac{GI_T}{EI_\omega} \qquad \beta = \frac{M_y^2}{EI_\omega \cdot EI_z}$$

$$\frac{d^4 \varphi}{d x^4} - 2\alpha \cdot \frac{d^2 \varphi}{d x^2} - \beta \cdot \varphi = 0$$

$$\varphi = A \cdot e^{m \cdot x}$$

$$A \cdot e^{m \cdot x} \cdot (m^4 - 2\alpha \cdot m^2 - \beta) = 0$$

Non-trivial solution:

$$m^4 - 2\alpha \cdot m^2 - \beta = 0$$

$$m = \pm\sqrt{\alpha \pm \sqrt{\beta + \alpha^2}}$$

$$\sqrt{\beta + \alpha^2} > \alpha$$

$$n^2 = \alpha + \sqrt{\beta + \alpha^2}$$

$$q^2 = -\alpha + \sqrt{\beta + \alpha^2}$$

$$\varphi = A_1 \cdot e^{n \cdot x} + A_2 \cdot e^{-n \cdot x} + A_3 \cdot e^{i \cdot q \cdot x} + A_4 \cdot e^{-i \cdot q \cdot x}$$

$$C_1 = 2 \cdot (A_1 + A_2) \qquad C_2 = 2 \cdot (A_1 - A_2)$$

$$C_3 = 2 \cdot i \cdot (A_3 - A_4) \qquad C_4 = 2 \cdot (A_3 + A_4)$$

$$\varphi = C_1 \cdot \operatorname{ch} nx + C_2 \cdot \operatorname{sh} nx + C_3 \cdot \sin qx + C_4 \cdot \cos qx$$

Boundary conditions:

$$\varphi(x=0) = \varphi(x=L) = \varphi''(x=0) = \varphi''(x=L) = 0$$

$$\downarrow$$

$$0 = C_1 \cdot 1 + C_2 \cdot 0 + C_3 \cdot 0 + C_4 \cdot 1;$$

$$0 = C_1 \cdot n^2 + C_2 \cdot 0 + C_3 \cdot 0 + C_4 \cdot (-q^2);$$

$$0 = C_1 \cdot \operatorname{ch} nL + C_2 \cdot \operatorname{sh} nL + C_3 \cdot \sin qL + C_4 \cdot \cos qL;$$

$$0 = C_1 \cdot n^2 \cdot \operatorname{ch} nL + C_2 \cdot n^2 \cdot \operatorname{sh} nL + C_3 \cdot (-q^2 \cdot \sin qL) + C_4 \cdot (-q^2 \cdot \cos qL).$$

$$\det\begin{bmatrix} 1 & 0 & 0 & 1 \\ n^2 & 0 & 0 & -q^2 \\ \operatorname{ch} nL & \operatorname{sh} nL & \sin qL & \cos qL \\ n^2\cdot \operatorname{ch} nL & n^2\cdot \operatorname{sh} nL & -q^2\cdot\sin qL & -q^2\cdot\cos qL \end{bmatrix} = 0$$

$$(n^2+q^2)^2\cdot \sin hnL\cdot \sin qL = 0$$

→ If: $n = 0$ $\qquad \operatorname{sh} nL = 0$

→ If: $n \neq 0$ $\qquad \sin qL = 0 \qquad q\cdot L = n\cdot\pi \qquad n = 1,2,...$

→ If: $n = 1$ $\qquad q = \sqrt{-\alpha+\sqrt{\beta+\alpha^2}} = \dfrac{\pi}{L}$

$$\frac{\pi^2}{L^2} = -\frac{GI_T}{2EI_\omega} + \sqrt{\frac{M_y^2}{EI_\omega\cdot EI_z} + \left(\frac{GI_T}{2EI_\omega}\right)^2}$$

$$M_{cr}^2 = \left[\left(\frac{\pi^2}{L^2} - \frac{GI_T}{2EI_\omega}\right)^2 - \left(\frac{GI_T}{2EI_\omega}\right)^2\right]\cdot EI_\omega\cdot EI_z$$

$$\boxed{M_{cr} = \frac{\pi}{L}\sqrt{EI_z\cdot\left(GI_T + EI_\omega\cdot\frac{\pi^2}{L^2}\right)}}$$

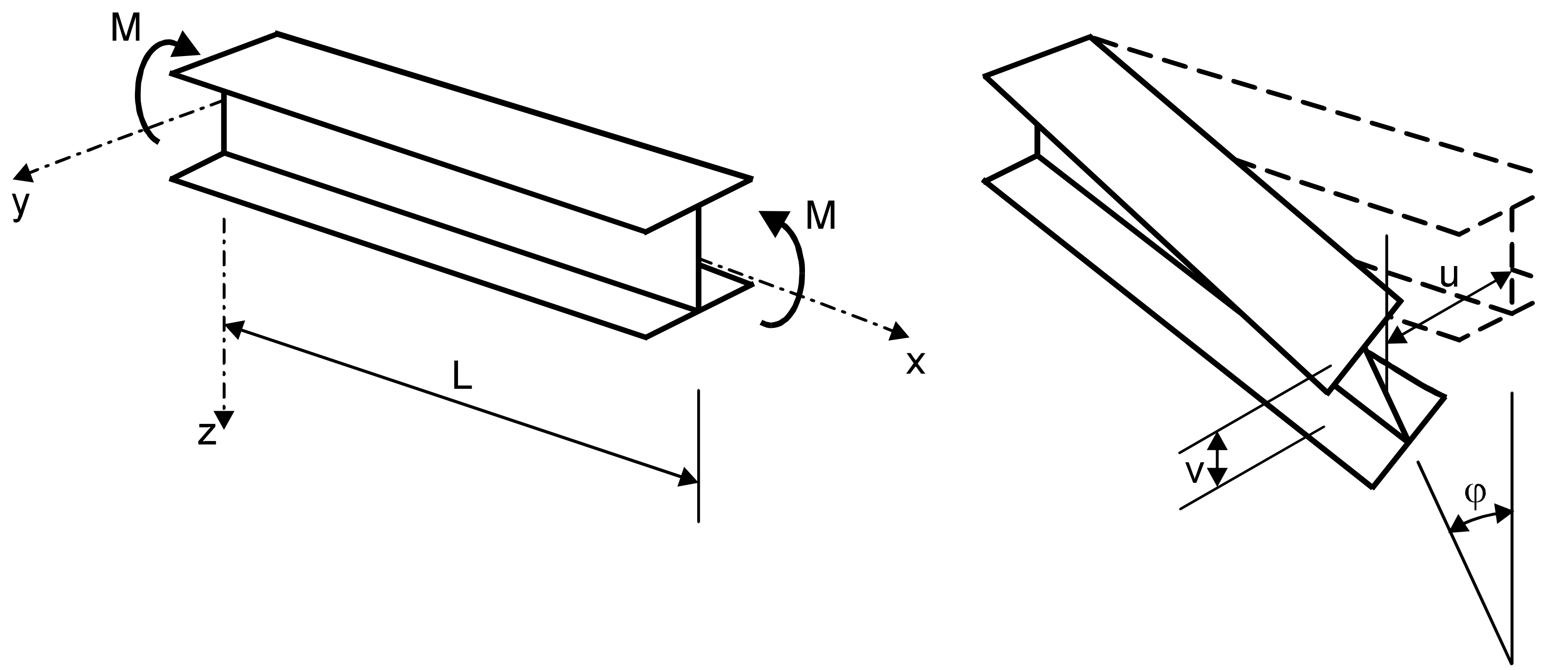
## 5.2.2 Energy Method for Beams in Bending

### *(a) "I" Cross-section Simple Supported Beams in Pure Bending*

[Chajes, 1974] [Allen, Bulson, 1980] [Chen, Lui, 1987]

Strain energy:

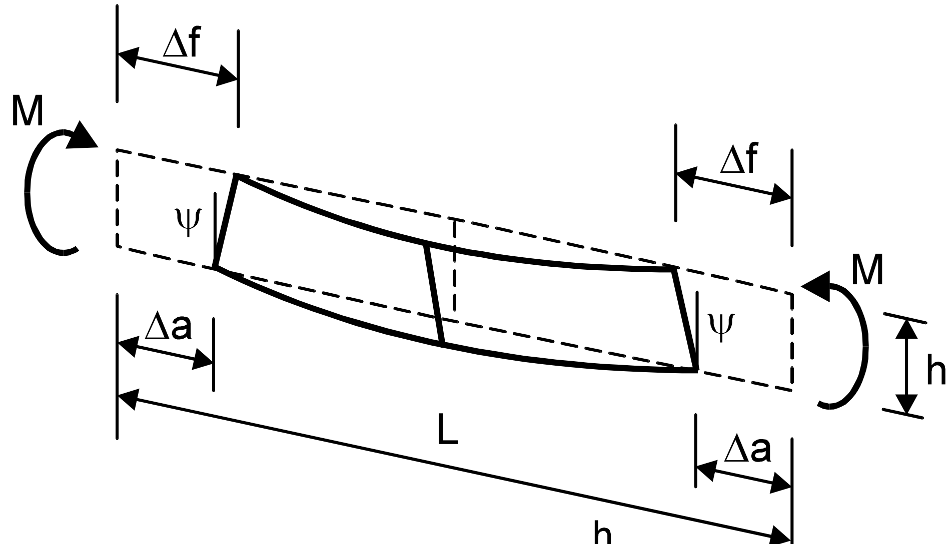
$$L_b = \frac{1}{2} \cdot EI_z \cdot \int_0^L \left( \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 \mathrm{d}x + \frac{1}{2} \cdot GI_T \cdot \int_0^L \left( \frac{\mathrm{d}\varphi}{\mathrm{d}x} \right)^2 \mathrm{d}x + \frac{1}{2} \cdot EI_\omega \cdot \int_0^L \left( \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} \right)^2 \mathrm{d}x$$



## Potential energy of external loads:

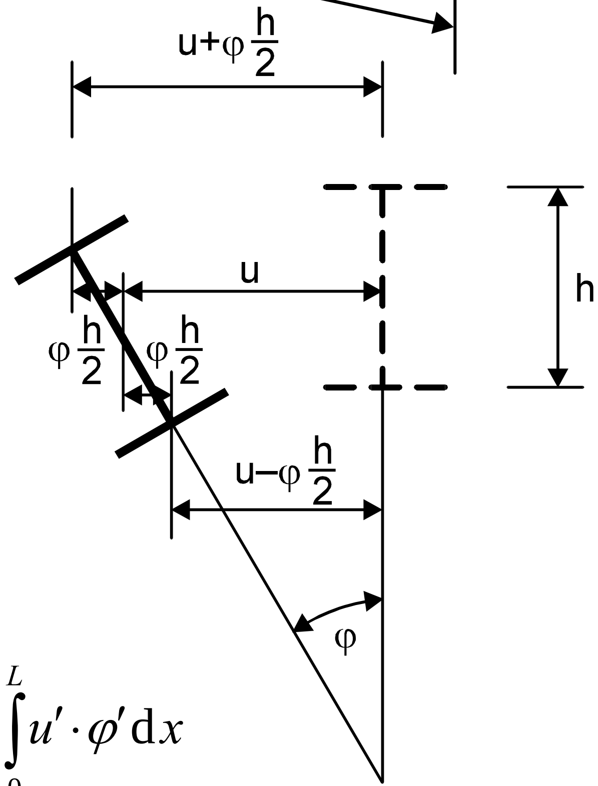
$$\boxed{L_k = -2M \cdot \psi}$$

$$\psi = \frac{\Delta f - \Delta a}{h}$$




$$\Delta f = \frac{1}{4} \cdot \int_0^L \left( \frac{\mathrm{d}\,u_f}{\mathrm{d}\,x} \right)^2 \mathrm{d}\,x \qquad \Delta a = \frac{1}{4} \cdot \int_0^L \left( \frac{\mathrm{d}\,u_a}{\mathrm{d}\,x} \right)^2 \mathrm{d}\,x$$

$$u_f = u + \varphi \cdot \frac{h}{2} \qquad u_a = u - \varphi \cdot \frac{h}{2}$$




$$\Delta f = \frac{1}{4} \cdot \int_0^L \left[ \frac{\mathrm{d}}{\mathrm{d}x} \left( u + \varphi \cdot \frac{h}{2} \right) \right]^2 \mathrm{d}\,x \qquad \Delta a = \frac{1}{4} \int_0^L \left[ \frac{\mathrm{d}}{\mathrm{d}x} \left( u - \varphi \cdot \frac{h}{2} \right) \right]^2 \mathrm{d}\,x$$

$$\psi = \frac{1}{2} \cdot \int_0^L \frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot \frac{\mathrm{d}\,\varphi}{\mathrm{d}\,x} \mathrm{d}\,x \qquad L_k = -M \cdot \int_0^L \frac{\mathrm{d}\,u}{\mathrm{d}\,x} \cdot \frac{\mathrm{d}\,\varphi}{\mathrm{d}\,x} \mathrm{d}\,x$$

$$\delta^2 \Pi = \frac{1}{2} \cdot EI_z \cdot \int_0^L (u'')^2 \,\mathrm{d}\,x + \frac{1}{2} \cdot GI_T \cdot \int_0^L (\varphi')^2 \,\mathrm{d}\,x + \frac{1}{2} \cdot EI_\omega \cdot \int_0^L (\varphi'')^2 \,\mathrm{d}\,x - M \cdot \int_0^L u' \cdot \varphi' \,\mathrm{d}\,x$$

## Boundary conditions:

$$x = 0,\quad x = L \;\longrightarrow\; u = v = u'' = 0,\quad v'' = \varphi = \varphi'' = 0$$

$$u = A \cdot \sin\frac{\pi x}{L} \qquad \varphi = B \cdot \sin\frac{\pi x}{L}$$

$$\varphi = -\frac{EI_z}{M} \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$$

$$A = B \cdot \frac{L^2}{\pi^2} \cdot \frac{M}{EI_z}$$

$$u = \frac{B \cdot L^2}{\pi^2} \cdot \frac{M}{EI_z} \cdot \sin\frac{\pi x}{L}$$

$$\delta^2\Pi = \frac{1}{2} \cdot \frac{B^2 \cdot M^2}{EI_z} \cdot \int_0^L \sin^2\frac{\pi x}{L}\,\mathrm{d}x + \frac{1}{2} \cdot GI_T \cdot B^2 \cdot \frac{\pi^2}{L^2} \cdot \int_0^L \cos^2\frac{\pi x}{L}\,\mathrm{d}x +$$
$$+ \frac{1}{2} \cdot EI_\omega \cdot B^2 \cdot \frac{\pi^4}{L^4} \cdot \int_0^L \sin^2\frac{\pi x}{L}\,\mathrm{d}x - \frac{M^2 \cdot B^2}{EI_z} \cdot \int_0^L \cos^2\frac{\pi x}{L}\,\mathrm{d}x.$$

$$\int_0^L \sin^2\frac{\pi x}{L}\,\mathrm{d}x = \int_0^L \cos^2\frac{\pi x}{L}\,\mathrm{d}x = \frac{L}{2}$$

$$\delta^2\Pi = \frac{1}{4} \cdot \left( \frac{GI_T \cdot B^2 \cdot \pi^2}{L} + \frac{EI_\omega \cdot B^2 \cdot \pi^4}{L^3} - \frac{M^2 \cdot B^2 \cdot L}{EI_z} \right)$$

$$\frac{B}{2} \cdot \left( \frac{GI_T \cdot \pi^2}{L} + \frac{EI_\omega \cdot \pi^4}{L^3} - \frac{M^2 \cdot L}{EI_z} \right) = 0$$

$$\frac{GI_T \cdot \pi^2}{L} + \frac{EI_\omega \cdot \pi^4}{L^3} - \frac{M^2 \cdot L}{EI_z} = 0$$

$$\boxed{M_{cr} = \frac{\pi}{L} \cdot \sqrt{EI_z \cdot \left( GI_T + EI_\omega \cdot \frac{\pi^2}{L^2} \right)}}$$

## (b) Fixed Ends Beam in Pure Bending

Boundary conditions:

$$x = 0,\quad x = L \;\rightarrow\; v = v'' = u = 0,\quad u' = \varphi = \varphi' = 0$$

$$u = A \cdot \left(1 - \cos\frac{2\pi x}{L}\right) \qquad \varphi = B \cdot \left(1 - \cos\frac{2\pi x}{L}\right)$$

$$\delta^2\Pi = \frac{1}{2} \cdot EI_z \cdot \frac{16A^2 \cdot \pi^4}{L^4} \cdot \int_0^L \cos^2\frac{2\pi x}{L}\,\mathrm{d}x + \frac{1}{2} \cdot GI_T \cdot \frac{4B^2 \cdot \pi^4}{L^2} \cdot \int_0^L \sin^2\frac{2\pi x}{L}\,\mathrm{d}x + \frac{1}{2} \cdot EI_\omega \cdot \frac{16B^2 \cdot \pi^4}{L^4} \cdot \int_0^L \cos^2\frac{2\pi x}{L}\,\mathrm{d}x - M \cdot \frac{4A \cdot B \cdot \pi^2}{L^2} \cdot \int_0^L \sin^2\frac{2\pi x}{L}\,\mathrm{d}x.$$

$$\int_0^L \sin^2\frac{2\pi x}{L}\,\mathrm{d}x = \int_0^L \cos^2\frac{2\pi x}{L}\,\mathrm{d}x = \frac{L}{2}$$

$$\delta^2\Pi = \frac{\pi^2}{L} \cdot \left(4EI_z \cdot \frac{A^2 \cdot \pi^2}{L^2} + GI_T \cdot B^2 + 4EI_\omega \cdot B^2 \cdot \frac{\pi^2}{L^2} - 2M \cdot A \cdot B\right)$$

$$\frac{\pi^2}{L} \cdot \left(8EI_z \cdot A \cdot \frac{\pi^2}{L^2} - 2M \cdot B\right) = 0$$

$$\frac{\pi^2}{L} \cdot \left(2GI_T \cdot B + 8EI_\omega \cdot B \cdot \frac{\pi^2}{L^2} - 2M \cdot A\right) = 0$$

$$4EI_z \cdot \frac{\pi^2}{L^2} \cdot \left(GI_T + 4EI_\omega \cdot \frac{\pi^2}{L^2}\right) - M^2 = 0$$

$$\boxed{M_{cr} = \frac{2\pi}{L} \cdot \sqrt{EI_z \cdot \left(GI_T + 4EI_\omega \cdot \frac{\pi^2}{L^2}\right)}}$$

$$\boxed{M_{cr} = \frac{\pi}{\nu_z \cdot L} \cdot \sqrt{EI_z \cdot \left(GI_T + EI_\omega \cdot \frac{\pi^2}{(\nu_\omega \cdot L)^2}\right)}}$$

$$\nu_z = \nu_\omega = 0.5$$

## (c) Uniform Bending – The Ends Free to Rotate About Horizontal and Vertical Axis, but Fully Restrained Against the Warping of End Cross-section

Boundary conditions:

$$x = 0,\; x = L \;\rightarrow\; v = v'' = u = 0,\quad u'' = \varphi = \varphi' = 0$$

$$u = A \cdot \sin\frac{\pi x}{L} \qquad \varphi = B \cdot \left(1 - \cos\frac{2\pi x}{L}\right)$$

$$\delta^2\Pi_1 = \frac{1}{2} \cdot EI_z \cdot \frac{A^2 \cdot \pi^4}{L^4} \cdot \int_0^L \sin^2\frac{\pi x}{L}\,\mathrm{d}x = \frac{1}{4} \cdot EI_z \cdot \frac{\pi^4}{L^3} \cdot A^2$$

$$\delta^2\Pi_2 = \frac{1}{2} \cdot GI_T \cdot \frac{4\pi^2}{L^2} \cdot \frac{L}{2} \cdot B^2 = \frac{\pi^2 \cdot GI_T}{L} \cdot B^2$$

$$\delta^2\Pi_3 = \frac{1}{2} \cdot EI_\omega \cdot \frac{16\pi^4}{L^4} \cdot \frac{L}{2} \cdot B^2 = \frac{4\pi^4 \cdot EI_\omega}{L^3} \cdot B^2$$

$$\delta^2\Pi_4 = -M \cdot A \cdot B \cdot \frac{2\pi^2}{L^2} \cdot \int_0^L \cos\frac{\pi x}{L}\sin\frac{2\pi x}{L}\,\mathrm{d}x = -\frac{8M \cdot \pi}{3L} \cdot A \cdot B$$

$$\delta^2\Pi = \frac{\pi^4 \cdot EI_z}{4L^3} \cdot A^2 - \frac{8M \cdot \pi}{3L} \cdot A \cdot B + \left(\frac{4\pi^4 \cdot EI_\omega}{L^3} + \frac{GI_T \cdot \pi^2}{L}\right) \cdot B^2$$

$$\frac{\pi^4 \cdot EI_z}{2L^3} \cdot A - \frac{8M \cdot \pi}{3L} \cdot B = 0$$

$$-\frac{8M \cdot \pi}{3L} \cdot A + \left(\frac{8\pi^4 \cdot EI_\omega}{L^3} + \frac{2GI_T \cdot \pi^2}{L}\right) \cdot B = 0$$

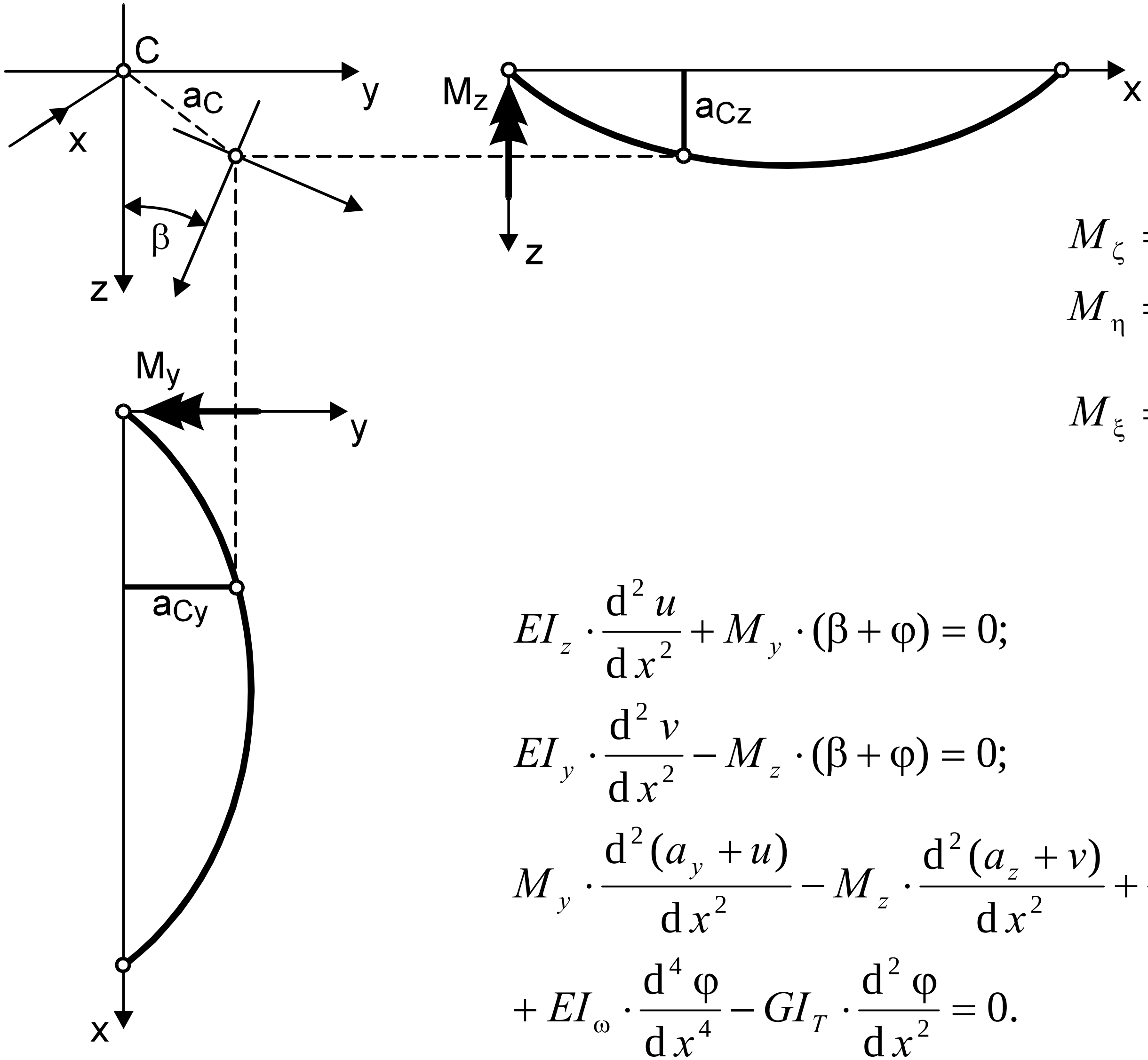
$$M_{cr} = \frac{\pi}{\nu_z \cdot L} \cdot \sqrt{EI_z \cdot \left(GI_T + EI_\omega \cdot \frac{\pi^2}{(\nu_\omega \cdot L)^2}\right)}$$

$$\nu_z = 0.85 \qquad \nu_\omega = 0.5$$

This value is 3.3% more than the “exact” result

# 5.2.3 Lateral-Torsional Buckling of Beams with Initial Imperfections

[Hunyadi, 1962]




$$M_\zeta = -M_y \cdot (\beta + \varphi);$$

$$M_\eta = -M_z \cdot (\beta + \varphi);$$

$$M_\xi = M_y \cdot \frac{\mathrm{d}(a_y + u)}{\mathrm{d}x} - M_z \cdot \frac{\mathrm{d}(a_z + v)}{\mathrm{d}x} + \\ + \frac{\mathrm{d}(\beta + \varphi)}{\mathrm{d}x} \cdot (M_y \cdot \beta_y + M_z \cdot \beta_z)$$

$$EI_z \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + M_y \cdot (\beta + \varphi) = 0;$$

$$EI_y \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - M_z \cdot (\beta + \varphi) = 0;$$

$$M_y \cdot \frac{\mathrm{d}^2 (a_y + u)}{\mathrm{d}x^2} - M_z \cdot \frac{\mathrm{d}^2 (a_z + v)}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 (\beta + \varphi)}{\mathrm{d}x^2} \cdot (M_y \cdot \beta_y + M_z \cdot \beta_z) + \\ + EI_\omega \cdot \frac{\mathrm{d}^4 \varphi}{\mathrm{d}x^4} - GI_T \cdot \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} = 0.$$

$$P_z \cdot A - M_y \cdot C = M_y \cdot C_0;$$
$$P_y \cdot B + M_z \cdot C = -M_z \cdot C_0;$$
$$-M_y \cdot A + M_z \cdot B + B_\omega - (M_y \cdot \beta_y + M_z \cdot \beta_z) \cdot C =$$
$$= M_y \cdot A_0 - M_z \cdot B_0 + (M_y \cdot \beta_y + M_z \cdot \beta_z) \cdot C_0.$$

$$\frac{A_0}{B_0} = \frac{A}{B} \qquad \frac{B_0}{C_0} = \frac{B}{C} \qquad \frac{A_0}{C_0} = \frac{A}{C}$$

$$\frac{A}{B} = -\frac{P_z \cdot M_{z,cr,1}}{P_y \cdot M_{y,cr,1}}$$

$$A = A_0 \cdot \frac{M}{M_{cr,1} - M};$$
$$B = B_0 \cdot \frac{M}{M_{cr,1} - M};$$
$$C = C_0 \cdot \frac{M}{M_{cr,1} - M}.$$

$$\frac{B}{C} = -\frac{B_\omega - (M_y \cdot \beta_y + M_z \cdot \beta_z)}{\dfrac{P_y \cdot M_{y,cr,1}^2}{P_z \cdot M_{z,cr,1}} + M_{z,cr,1}}$$

$$\frac{A}{C} = -\frac{B_\omega - (M_y \cdot \beta_y + M_z \cdot \beta_z)}{\dfrac{P_z \cdot M_{z,cr,1}^2}{P_y \cdot M_{y,cr,1}} + M_{y,cr,1}}$$

$B_\omega$ – for open cross-section

$B_\omega^*$ – for closed cross-section

# 5.2.4 Design Method: Fundamental Solutions

Two components of vector of bending moments ***M*** :

$$M_y = M \cdot \cos\alpha \qquad M_z = M \cdot \sin\alpha$$

Bending moments:

$$M_\eta = -EI_y \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + M_y;$$

$$M_\zeta = -EI_z \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + M_z;$$

Bimoment:

$$B = -EI_\omega \cdot \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2}$$

Stresses:

$$\sigma_y = \frac{M_y}{I_y} \cdot z + E \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \cdot z;$$

$$\sigma_z = \frac{M_z}{I_z} \cdot y + E \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \cdot y;$$

$$\sigma_\omega = E \cdot \frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} \cdot \omega.$$

$$f_y \geq \sigma \cdot \left( 1 + \frac{\pi^2 \cdot E}{L^2} \cdot \frac{\Omega}{\sigma_{kr,1} - \sigma} \right)$$

$$\sigma_H \geq \sigma \cdot \left( 1 + \frac{\pi^2 \cdot E \cdot \sigma_H}{f_y \cdot L^2} \cdot \frac{\Omega}{\frac{\sigma_H}{f_y} \cdot \sigma_{kr,1} - \sigma} \right)$$

$$\sigma = M / W \qquad \sigma_{kr,1} = M_{kr,1} / W$$

Superpose:

$$\frac{M}{W} = \frac{M_y}{W_y} + \frac{M_z}{W_z}$$

$$W = \frac{M \cdot W_y \cdot W_z}{M_y \cdot W_z + M_z \cdot W_y}$$

Open cross-section: $\sigma_\omega = E \cdot \dfrac{d^2 \varphi}{d x^2} \cdot \omega$

Closed cross-section: $\sigma^*_\omega = E \cdot \dfrac{d^2 \phi}{d x^2} \cdot \omega^*$

$$G \cdot (I_T^* - I_p) \cdot \frac{d\phi}{dx} + GI_p \cdot \frac{d\varphi}{dx} = M_\xi$$

$$\frac{d^2 \phi}{d x^2} = \frac{1}{\mu} \cdot \frac{d^2 \varphi}{d x^2} - \frac{1}{\mu \cdot GI_p} \cdot \frac{d M_\xi}{d x}$$

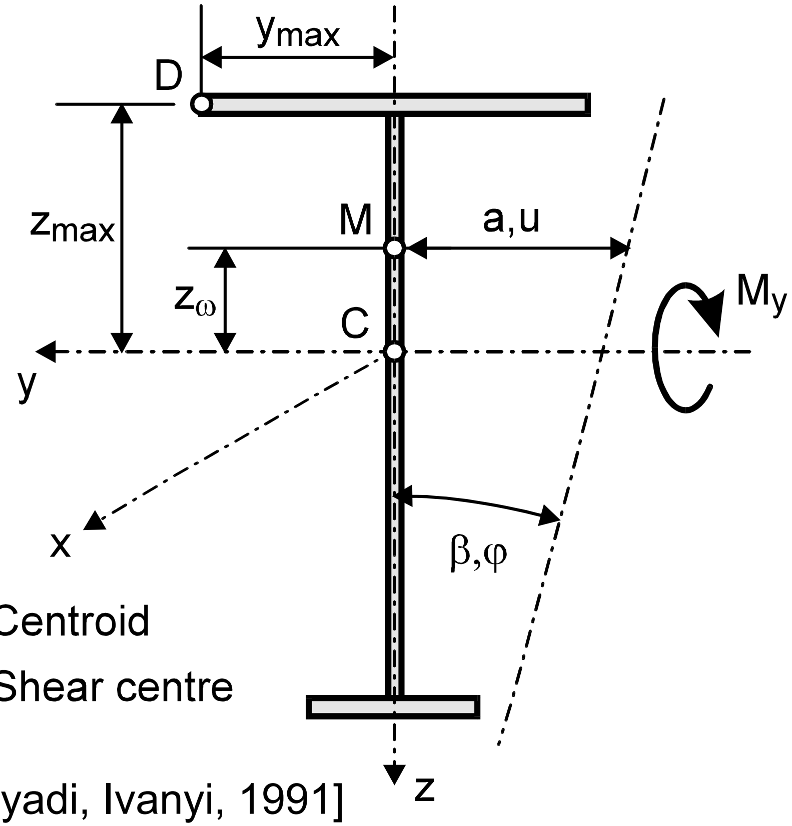
$$\phi = g \cdot \sin\frac{\pi}{L}x \qquad \varphi = C \cdot \sin\frac{\pi}{L}x$$

$$\phi = g \cdot \sin\frac{\pi}{L}x \qquad g = \frac{\mu}{\rho} \cdot C$$

$$\Omega^* = A_0 \cdot y + B_0 \cdot z + \frac{\mu}{\rho} \cdot C_0 \cdot \omega^*$$

$$\Omega^* = \nu \cdot L^2$$

*(a) Mono-Symmetric "I" Cross-section*




C – Centroid

M – Shear centre

[Hunyadi, Ivanyi, 1991]

Equilibrium DE:

$$EI_z \cdot u'' + M \cdot (\beta + \varphi) = 0;$$
$$EI_\omega \cdot \varphi^{IV} - GI_T \cdot \varphi'' + M \cdot (a'' + u'') +$$
$$+ M \cdot f_y \cdot (\beta'' + \varphi'') = 0,$$

$$u = A \cdot \sin\frac{\pi}{L}x \qquad \varphi = C \cdot \sin\frac{\pi}{L}x$$

$$a = A_0 \cdot \sin\frac{\pi}{L}x \qquad \beta = C_0 \cdot \sin\frac{\pi}{L}x$$

$$P_z = \frac{\pi^2 \cdot EI_z}{L^2} \qquad B_\omega = \frac{\pi^2 \cdot EI_\omega}{L^2} + GI_T$$

$$P_z \cdot A - M \cdot C = M \cdot C_0;$$
$$-M \cdot A + (B_\omega - M \cdot \beta_y) \cdot C = M \cdot (A_0 + \beta_y \cdot C_0)$$

$$A = \frac{1}{D} \cdot (C_0 \cdot M \cdot B_\omega + A_0 \cdot M^2);$$
$$C = \frac{1}{D} \cdot (A_0 \cdot M \cdot P_z + C_0 \cdot M^2 + C_0 \cdot P_z \cdot M \cdot \beta_y),$$

$$D = P_z \cdot (B_\omega - M \cdot \beta_y) - M^2$$

$$r = \frac{M \cdot (M \cdot A_0 + B_\omega \cdot C_0)}{M \cdot \left[M \cdot C_0 + P_z \cdot (C_0 \cdot \beta_y + A_0)\right]}$$

At the beginning of buckling: $r = A_0 / C_0$

$$r = \sqrt{\frac{B_\omega}{P_z} + \frac{\beta_y}{2}} - \frac{\beta_y}{2}$$

Critical moment: $D = 0$

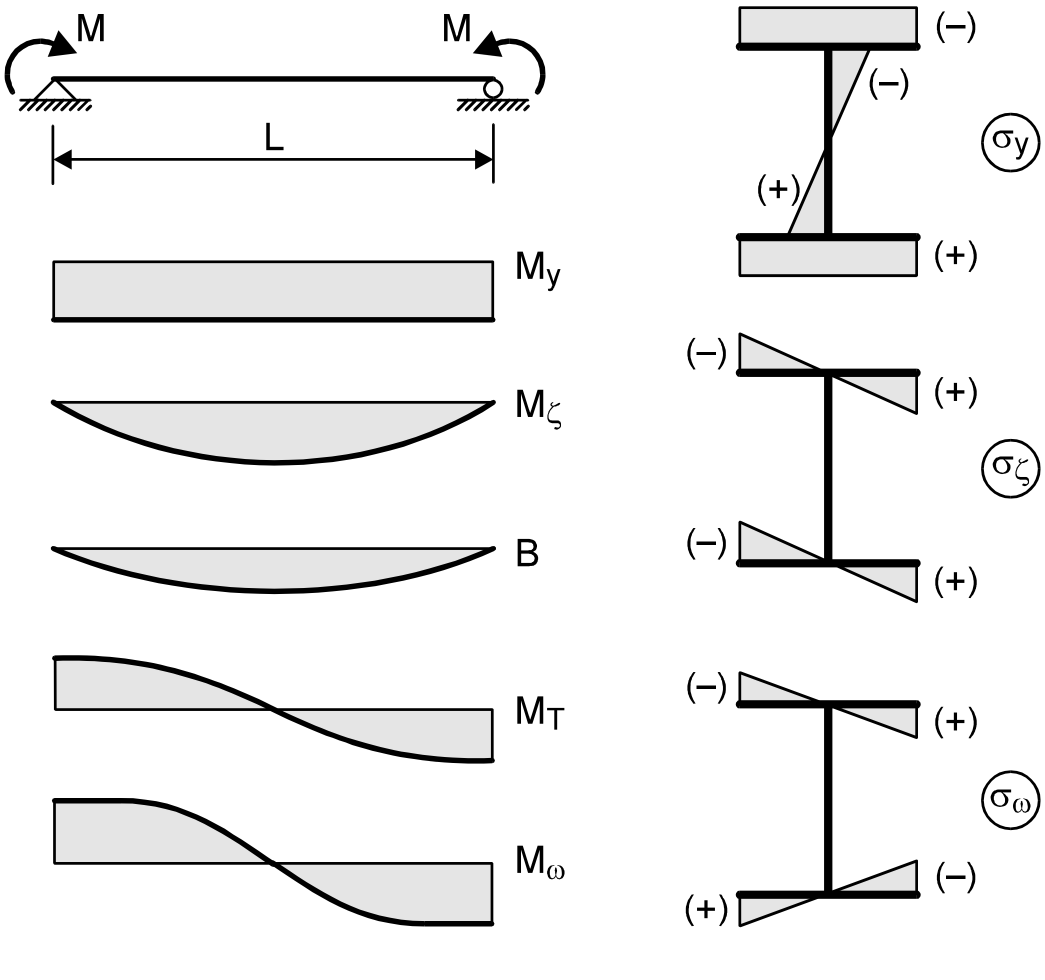
$$M_{cr,1,2} = \pm\sqrt{B_\omega \cdot P_z + \left(\frac{P_z \cdot \beta_y}{2}\right)^2} - \frac{P_z \cdot \beta_y}{2}$$

$$A = \frac{M \cdot A_0 \cdot \left(M + \frac{B_\omega}{r}\right)}{-(M_{cr,1} - M) \cdot (M_{cr,2} - M)}$$

$$M_{cr,1} = P_z \cdot r \qquad M_{cr,2} = -B_\omega / r$$

$$A = \frac{A_0}{\frac{M_{cr,1}}{M} - 1} \qquad C = \frac{C_0}{\frac{M_{cr,1}}{M} - 1}$$

Safety factor: $n = f_y / \sigma_H$




(a) (b)

$$\sigma = \frac{M}{I_y} \cdot z_{\max} \qquad \sigma_1 = \frac{M_1}{I_y} \cdot z_{\max}$$

$$\varphi_k = \frac{\sigma}{\sigma_H} \qquad \lambda_1^2 = \frac{\sigma_1}{f_y}$$

$$\lambda_E^2 = \frac{\pi^2 \cdot E}{f_y}$$

$$\varphi_k = \beta_1 - \sqrt{\beta_1^2 - \lambda_1^2}$$

$$\beta_1 = \frac{1}{2} \cdot \left( 1 + \lambda_1^2 + \lambda_E^2 \cdot \frac{\Omega}{L^2} \right)$$

$$\Omega = A_0 \cdot y_{\max} + C_0 \cdot \omega_{\max}$$

$$\sum \sigma = \sigma + E \cdot y_{\max} \cdot u'' + E \cdot \omega_{\max} \cdot \varphi''$$

$$f_y \geq n \cdot \frac{M_y}{I_y} \cdot z_{\max} + \frac{E \cdot \pi^2}{L^2} \cdot \frac{A_0 \cdot y_{\max}}{\frac{M_{cr,1}}{n \cdot M} - 1} + \frac{E \cdot \pi^2}{L^2} \cdot \frac{C_0 \cdot \omega_{\max}}{\frac{M_{cr,1}}{n \cdot M} - 1}$$

In the previous figure:

$$\omega_{\max} = (z_{\max} - z_{\omega}) \cdot y_{\max}$$

$$\Omega = A_{D0} \cdot y_{\max}$$

$$A_{D0} = k \cdot \alpha \cdot (\bar{\lambda} - 0{,}2)$$

$$\bar{\lambda} = \sqrt{f_y / \sigma_{cr}}$$

At lateral-torsional buckling: $\sigma_{cr} = \sigma_1$

$$A_{D0} = \alpha \cdot k_z \cdot \left( \frac{1}{\lambda_1} - 0{,}2 \right) \cdot \frac{r}{k_y}$$

$$y_{\max} \cdot k_z = i_z^2$$

$$\beta_1 = \frac{1}{2} \cdot \left[ 1 + \lambda_1^2 + \frac{\lambda_E^2}{\lambda_1^2} \cdot \alpha \cdot \left( \frac{1}{\lambda_1} - 0{,}2 \right) \right]$$

For double-symmetric cross-section it can be supposed: $A_0 = 0$

Thus:

$$D = P_z \cdot B_\omega - M^2 = M_{cr}^2 - M^2$$

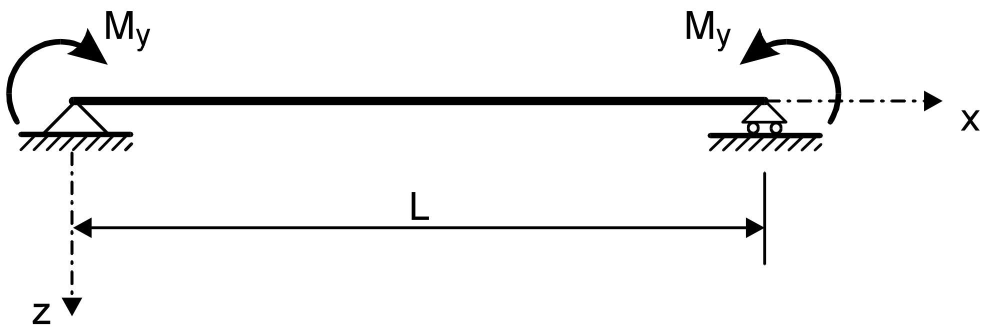
$$C = \frac{1}{D} \cdot C_0 \cdot M^2$$

$$C = \frac{M^2}{M_{cr}^2 - M^2} \cdot C_0 = \frac{\left( \frac{M}{M_{cr}} \right)^2}{1 - \left( \frac{M}{M_{cr}} \right)^2} \cdot C_0$$

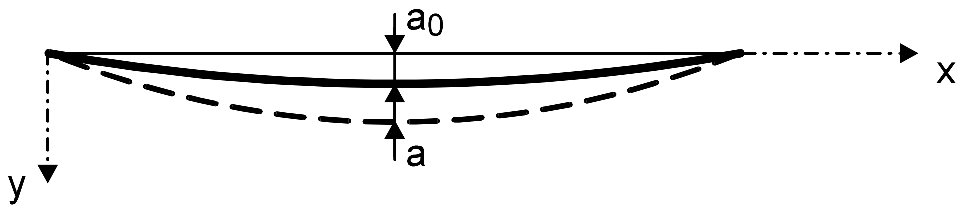
$$C_1 = C + C_0 = \frac{1}{1 - \left( \frac{M}{M_{cr}} \right)^2} \cdot C_0$$

## *(b) Ayrton-Perry Formula*

[Costa Ferreira, Rondal, 1987]




Initial imperfection: $\varphi_0 = a_0 \cdot \sin \frac{\pi x}{L}$




$$\varphi = \varphi_0 \cdot \frac{1}{1 - \frac{M_y^2}{M_{cr}^2}} \qquad M_z = M_y \cdot \varphi$$

$$\frac{M_y}{W_y} + \frac{M_z}{W_z} = f_y \qquad \frac{M_y}{W_y} + \frac{M_y \cdot \varphi}{W_z} = f_y$$

$$\frac{M_y}{W_y} + \frac{M_y}{W_y} \cdot \frac{W_y \cdot \varphi_0}{W_z} \cdot \frac{1}{1 - \left( \frac{M_y}{M_{cr}} \right)^2} = f_y$$

$$\frac{M_y}{W_y} \cdot \frac{W_y \cdot \varphi_0}{W_z} \cdot \frac{M_{cr}^2}{M_{cr}^2 - M_y^2} = f_y - \frac{M_y}{W_y}$$

$$\sigma_K = \frac{M_y}{W_y} \qquad \sigma_{cr} = \frac{M_{cr}}{W_y}$$

$$\sigma_K \cdot \frac{W_y \cdot \varphi_0}{W_z} \cdot \frac{\sigma_{cr}^2}{\sigma_{cr}^2 - \sigma_K^2} = f_y - \sigma_K$$

$$(f_y - \sigma_K)\cdot(\sigma_{cr}^2 - \sigma_K^2) = \frac{W_y \cdot \varphi_0}{W_z}\cdot \sigma_K \cdot \sigma_{cr}^2$$

$$\eta = W_y \varphi_0 / W_z$$

$$(f_y - \sigma_K)\cdot(\sigma_{cr}^2 - \sigma_K^2) = \eta \cdot \sigma_K \cdot \sigma_{cr}^2$$

$$\frac{\sigma_K}{f_y} = \overline{N}$$

$$\frac{f_y}{\sigma_{cr}} = \frac{\lambda^2}{\lambda_E^2} = \overline{\lambda}^2$$

$$(1-\overline{N})\cdot(1-\overline{N}^2\cdot\overline{\lambda}^4) = \eta \cdot \overline{N}$$

[Barta, 1972]

## Suggestion for the formula:

*(i) Costa Ferreira and Rondal I.*

$$\eta = \alpha \cdot (\overline{\lambda} - \overline{\lambda}_0) \qquad \overline{\lambda}_0 = 0.4$$

| Cross-section | $\alpha$ |
|---|---|
| Welded | 0.60 |
| Rolled | 1.20 |

*(ii) Costa Ferreira and Rondal II.*

$$(1-\overline{N})\cdot(1-\overline{N}\cdot\overline{\lambda}^2) = \eta \cdot \overline{N}$$

| Cross-section | $\alpha$ |
|---|---|
| Welded | 0.32 |
| Rolled | 0.78 |

Készült az ERFP – DD2002 – HU – B – 01 szerzősésszámú projekt támogatásával

## (c) Rankine-Merchant Formula

In elastic state:

$$M_y \cdot \Phi_z = \Phi_x \cdot K_x;$$
$$M_y = \Phi_y \cdot K_y;$$
$$M_y \cdot \Phi_x = (\Phi_z - \Phi_{z0}) \cdot K_z,$$

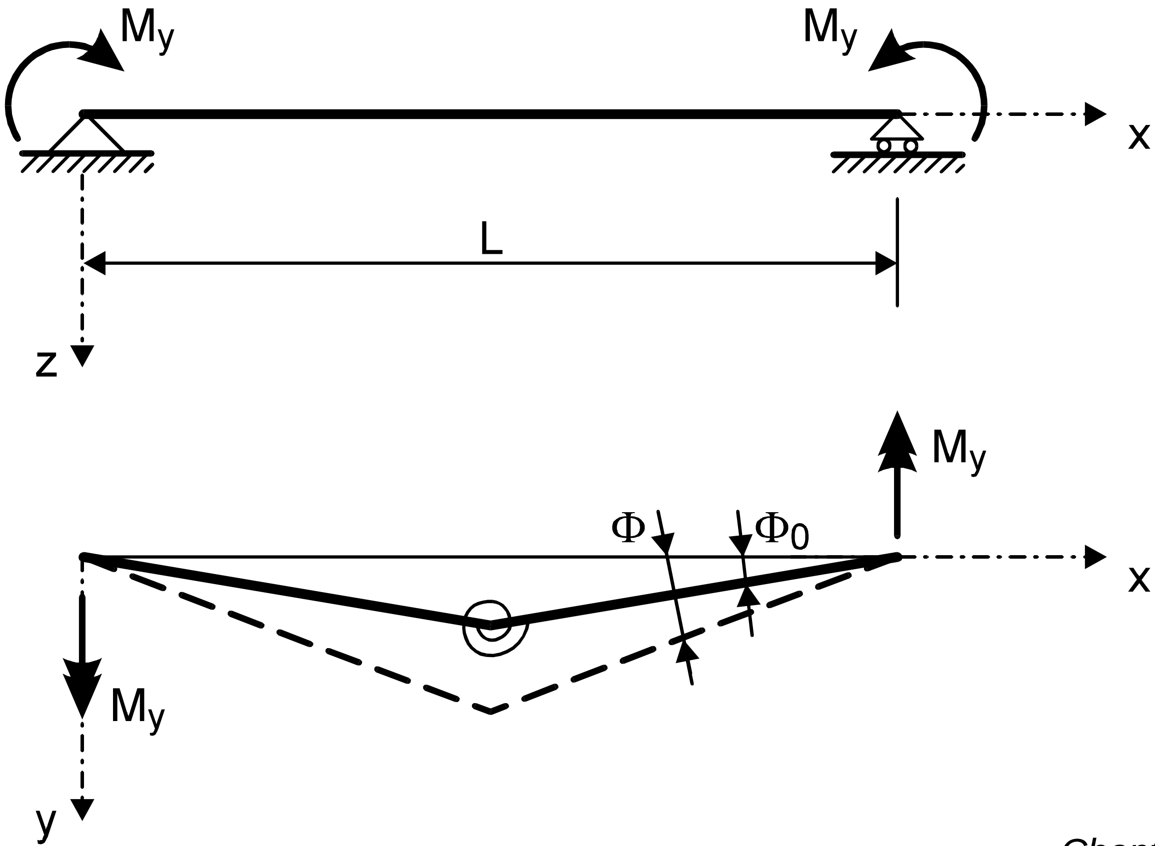
$$\Phi_z = \Phi_{z0} \cdot \frac{1}{1 - \frac{M_y^2}{M_{cr}^2}}$$

In plastic state:

$$M_y \cdot \Phi_z = M_{xpl};$$
$$M_y = M_{ypl};$$
$$M_y \cdot \Phi_x = M_{zpl}.$$

$$M_y^2 = \frac{M_{xpl} \cdot M_{zpl}}{\Phi_x \cdot \Phi_z}$$

$$\frac{1}{\sigma_K} = \frac{1}{\sigma_{pl}} + \frac{1}{\sigma_{cr}}$$



Expression:

$$M_{yK}^2 = \frac{M_{xpl} \cdot M_{zpl}}{\Phi_x \cdot \Phi_{z0} \cdot \dfrac{M_{cr}^2}{M_{yK}^2 - M_{cr}^2}} =$$

$$= \frac{M_{xpl} \cdot M_{zpl}}{\Phi_x \cdot \Phi_{z0} + \dfrac{M_{xpl} \cdot M_{zpl}}{M_{cr}^2}} =$$

$$= \frac{1}{\dfrac{\Phi_x \cdot \Phi_{z0}}{M_{xpl} \cdot M_{zpl}} + \dfrac{1}{M_{cr}^2}} = \frac{1}{\dfrac{1}{M_{pl}^2} + \dfrac{1}{M_{cr}^2}};$$

Result:

$$\frac{1}{M_{yK}^2} = \frac{1}{M_{pl}^2} + \frac{1}{M_{cr}^2}$$

$$\boxed{\frac{1}{\sigma_K^2} = \frac{1}{\sigma_{pl}^2} + \frac{1}{\sigma_{cr}^2}}$$

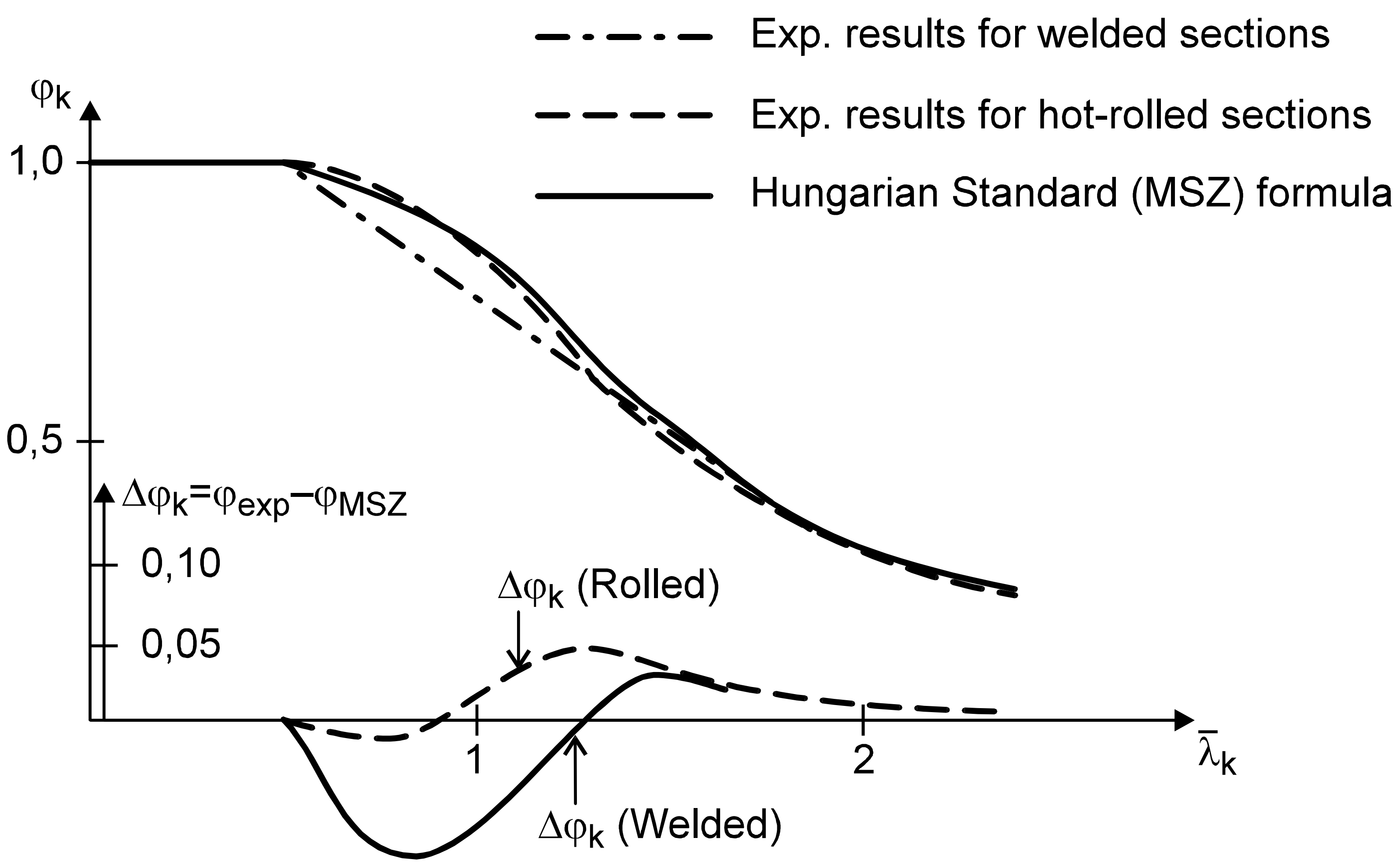
In general:

$$\boxed{\frac{1}{\sigma_K^n} = \frac{1}{\sigma_{pl}^n} + \frac{1}{\sigma_{cr}^n}}$$

*Fukumoto and Kubo Suggestion*

| |
|---|
| n=2.5 for rolled cross-sections |
| n=2.0 for welded cross-sections |

*(d) Experimental Tests and Hungarian Standard (MSZ)*



Experimental results for welded cross-sections:

$$\varphi_{\exp} = \begin{cases} 1{,}0 & \text{ha} \quad \bar{\lambda}_k \le 0{,}4 \\ -0{,}3\bar{\lambda}_k^2 + 0{,}0513\bar{\lambda}_k + 1{,}028 & \text{ha} \quad 0{,}4 \le \bar{\lambda}_k \le \sqrt{2} \\ \dfrac{1}{\bar{\lambda}_k^2} & \text{ha} \quad \sqrt{2} \le \bar{\lambda}_k \end{cases}$$

Experimental results for hot-rolled cross-sections:

$$\varphi_{\exp} = \begin{cases} 1{,}0 & \text{ha} \quad \bar{\lambda}_k \le 0{,}4 \\ -0{,}493\bar{\lambda}_k + 1{,}197 & \text{ha} \quad 0{,}4 \le \bar{\lambda}_k \le \sqrt{2} \\ \dfrac{1}{\bar{\lambda}_k^2} & \text{ha} \quad \sqrt{2} \le \bar{\lambda}_k \end{cases}$$

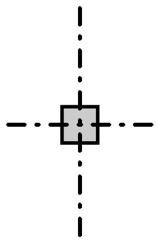
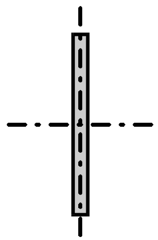
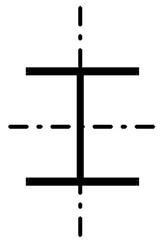
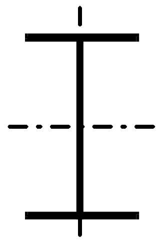
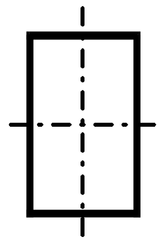
Hungarian Standard (MSZ) regulation:

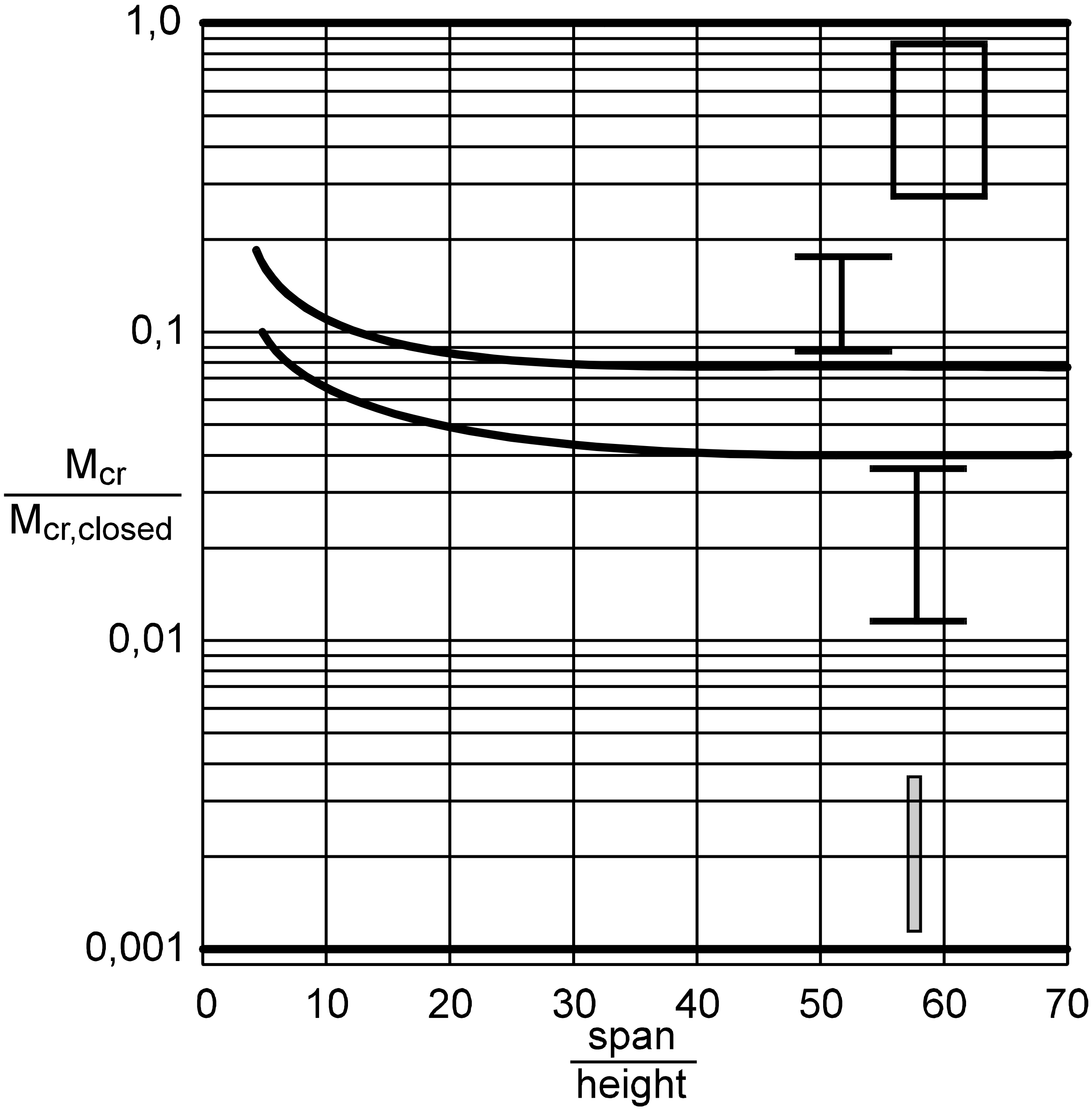
$$\varphi_k = \left( \frac{1}{1 + \bar{\lambda}_k^5} \right)^{0{,}4}$$

## *(e) Effect of the Shape of Cross-section*

[Trahair, 1993]

$$M_{cr} = \frac{\pi}{L} \cdot \sqrt{\frac{EI_z \cdot GI_T}{\gamma}} \cdot \sqrt{1 + \frac{\pi^2 \cdot EI_\omega}{L^2 \cdot GI_T}} \qquad \gamma = \frac{I_y - I_z}{I_y}$$

| Property |  |  |  |  |  |
|---|---|---|---|---|---|
| A | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $I_y$ | 1.0 | 25.0 | 12.45 | 45.59 | 16.94 |
| $I_z$ | 1.0 | 0.04 | 3.20 | 3.20 | 8.10 |
| $I_T$ | 1.0 | 0.04 | 0.034 | 0.033 | 4.731 |


1,0
0,1
0,01
0,001
Mcr
Mcr,closed
0
10
20
30
40
50
60
70
span
height

# 5.2.5 Design Method: General Solution

[Clark, Hill, 1961]

Total potential energy:

$$\delta^2\Pi = \frac{1}{2}\cdot\int_0^L \Big[EI_z\cdot(u'')^2 + EI_\omega\cdot(\varphi'')^2 + GI_T\cdot(\varphi')^2 + 2M\cdot\varphi\cdot u'' + + 2\beta_y\cdot M\cdot(\varphi')^2 + \bar{d}\cdot p\cdot\varphi^2\Big]\mathrm{d}\,x.$$

From the equilibrium:

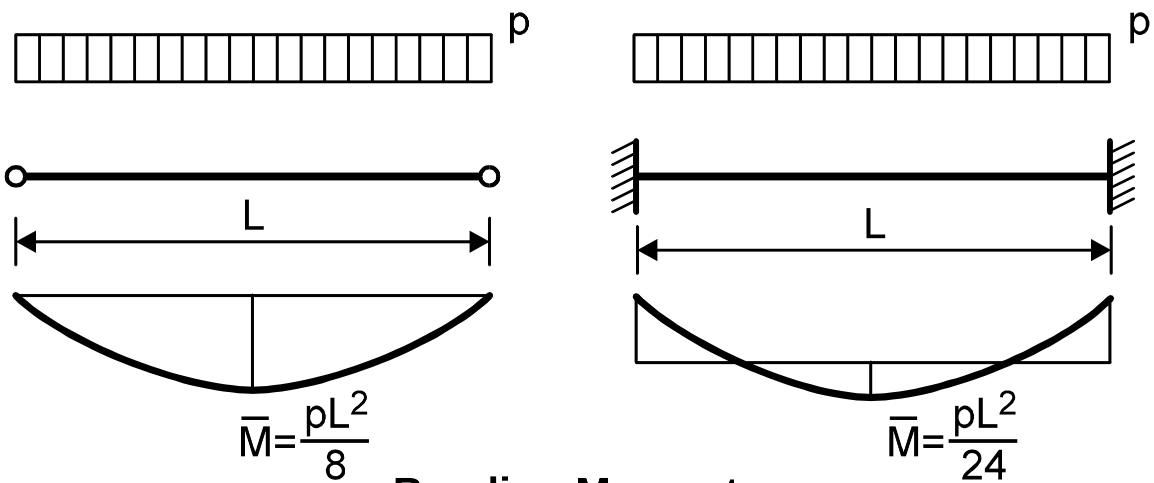
$$EI_z\cdot u'' + M\cdot\varphi = 0$$

Transversal load:

$$p = M''$$

Transformation:

$$X = \frac{x}{L} \qquad m = \frac{M(x)}{\overline{M}}$$




**Bending Moments**

$$\frac{\overline{M}^2}{EI_z}\cdot\int_0^1 m^2\cdot\varphi^2\cdot\mathrm{d}\,X - \frac{\overline{M}}{L^2}\cdot\left[\bar{d}\cdot\int_0^1\frac{\mathrm{d}^2 m}{\mathrm{d}\,X^2}\cdot\varphi^2\cdot\mathrm{d}\,X + 2\beta_y\cdot\int_0^1 m\cdot\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\,X}\right)^2\mathrm{d}\,X\right] -$$

$$-\frac{GI_T}{L^2}\cdot\int_0^1\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\,X}\right)^2\mathrm{d}\,X - \frac{EI_\omega}{L^4}\cdot\int_0^1\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}\,X^2}\right)^2\mathrm{d}\,X = 0.$$

(Eurocode 3)

$$I=\sqrt{\int_0^1 m^2\cdot\varphi^2\,\mathrm{d}X\cdot\int_0^1\frac{\mathrm{d}^2\varphi}{\mathrm{d}X^2}\mathrm{d}X}$$

$$C_1=\frac{\int_0^1\left(\frac{\mathrm{d}\varphi}{\mathrm{d}X}\right)^2\mathrm{d}X}{I}$$

$$C_2=-\frac{1}{2}\cdot\frac{\int_0^1\frac{\mathrm{d}^2 m}{\mathrm{d}X^2}\cdot\varphi^2\,\mathrm{d}X}{I}$$

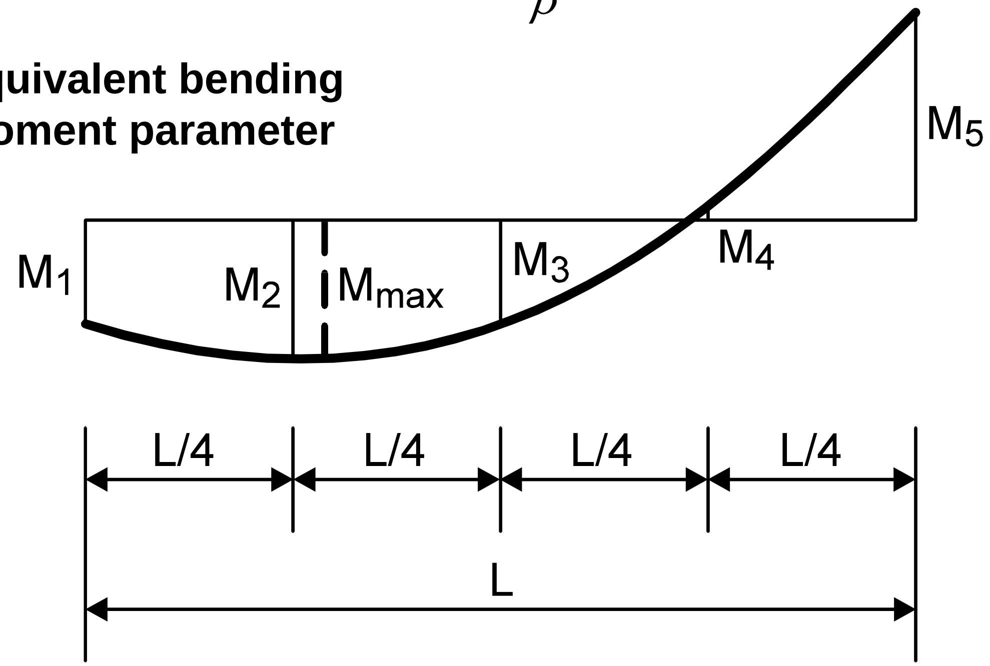
$$C_3=\frac{\int_0^1 m\cdot\left(\frac{\mathrm{d}\varphi}{\mathrm{d}X}\right)^2\mathrm{d}X}{I}$$

$$k^2=\pi^2\cdot\frac{\int_0^1\left(\frac{\mathrm{d}\varphi}{\mathrm{d}X}\right)^2\mathrm{d}X}{\int_0^1\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}X^2}\right)^2\mathrm{d}X}$$

$$\overline{M}_{cr}=C_1\cdot\frac{\pi^2\cdot EI_z}{(k\cdot L)^2}\cdot\left[C_2\cdot\bar{d}+C_3\cdot\beta_y+\sqrt{(C_2\cdot\bar{d}+C_3\cdot\beta_y)^2+\frac{I_\omega}{I_z}\cdot\left(1+\frac{GI_T\cdot(k\cdot L)^2}{\pi^2\cdot EI_\omega}\right)}\right]$$

$$\beta=M_e/M_{\max}\qquad C_1=\frac{1}{\beta}$$

**Equivalent bending moment parameter**

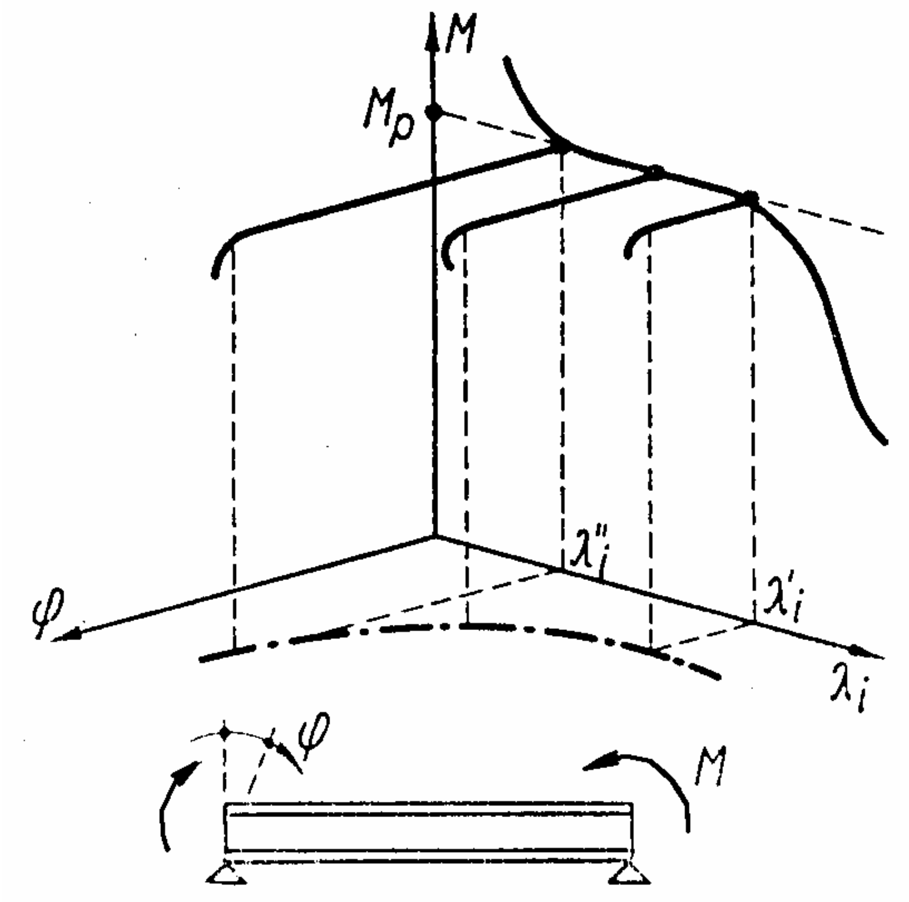



$$\beta=M_e/M_{\max}\qquad \beta=\frac{3M_2+4M_3+3M_4+2M_{\max}}{12M_{\max}}$$

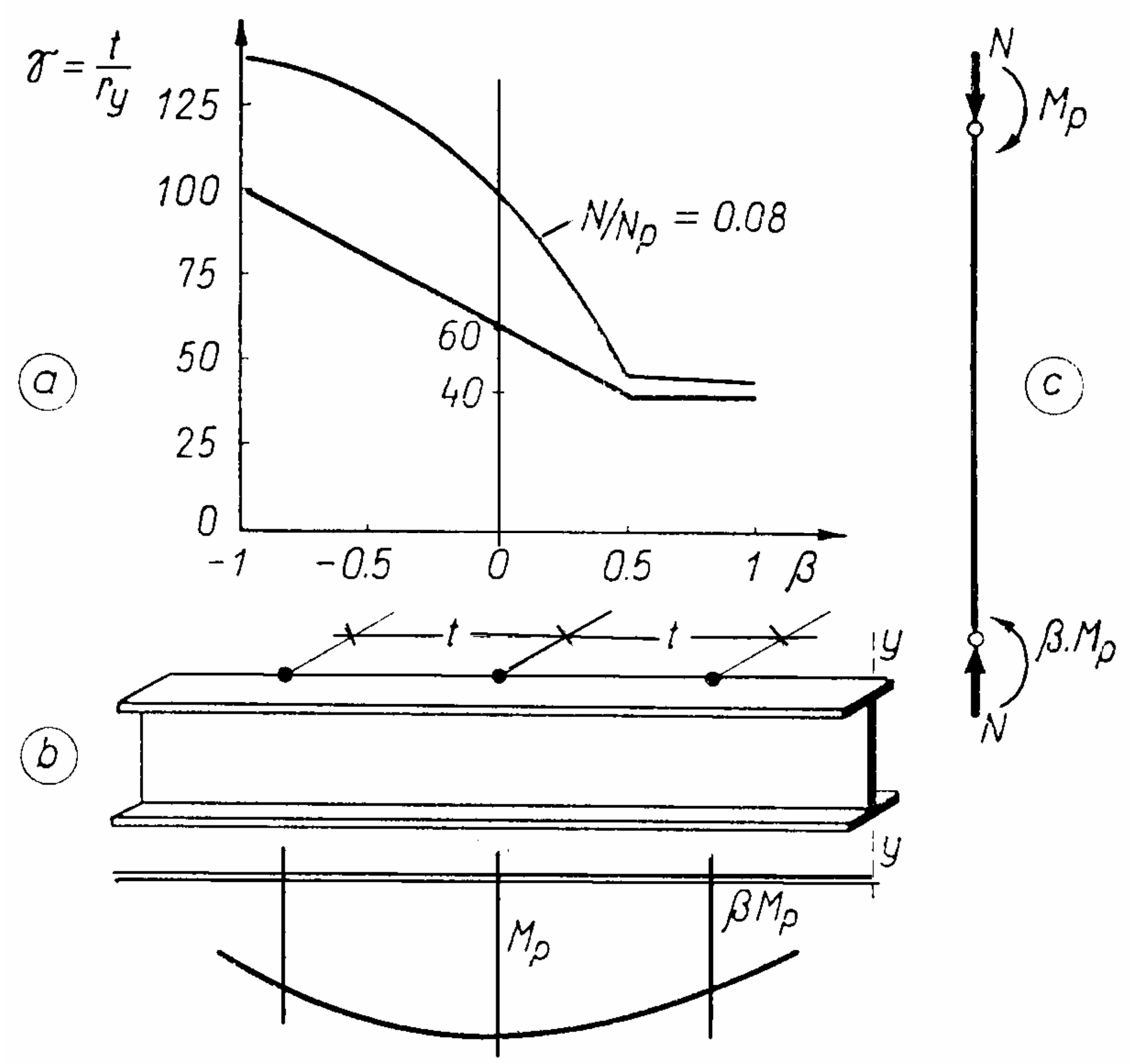
# 5.3. Lateral-Torsional Buckling of Beam-Columns

## 5.3.1 Stability Requirements. Experimental Results

### *5.3.1.1 General remarks*



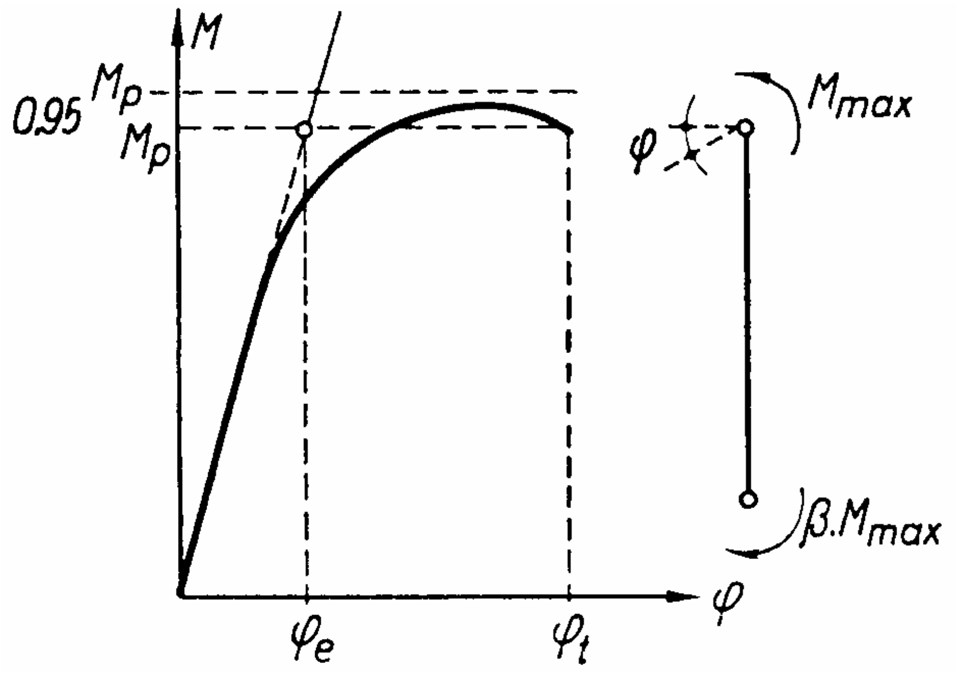
## 5.3.1.2 Bracing requirements for continuous beams

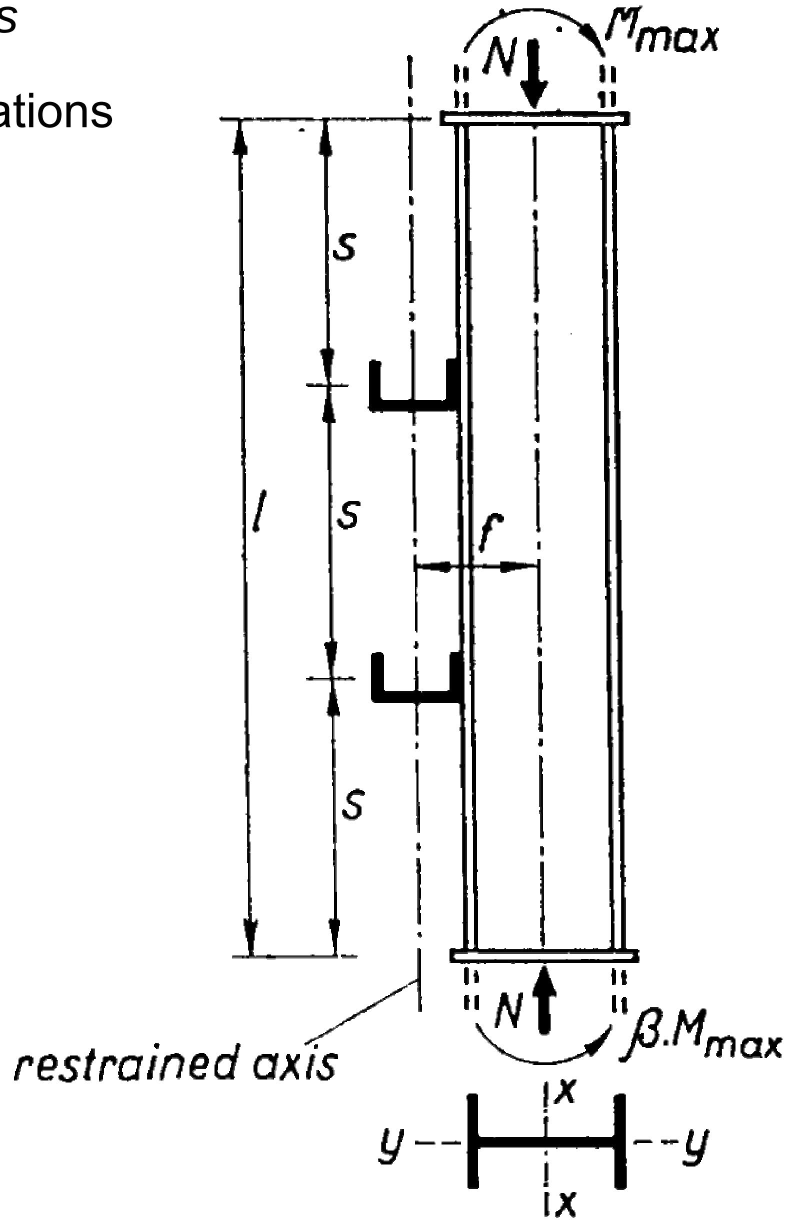


### *5.3.1.3 Bracing requirements for beam-columns*

(a) Requirements in the ECCS Recommendations

(b) Proposals of British authors



$$\mu = 0.6 + 0.4\beta; \qquad \text{if} \qquad -0.5 \leq \beta \leq 1;$$

$$\mu = 0.4; \qquad \text{if} \qquad -1 \leq \beta \leq -0.5.$$

$$\alpha = \frac{G \cdot K - N(r_x^2 + r_y^2 + f^2)}{\dfrac{\pi^2 E J_y}{l^2}\left(\dfrac{D^2}{4} + f^2\right)} \cdot D^2.$$

In this case the $\mu$ value can be approximated by the formula

$$\mu = A(\alpha) + B(\alpha) \cdot \beta + C(\alpha) \cdot \beta^2$$

where:

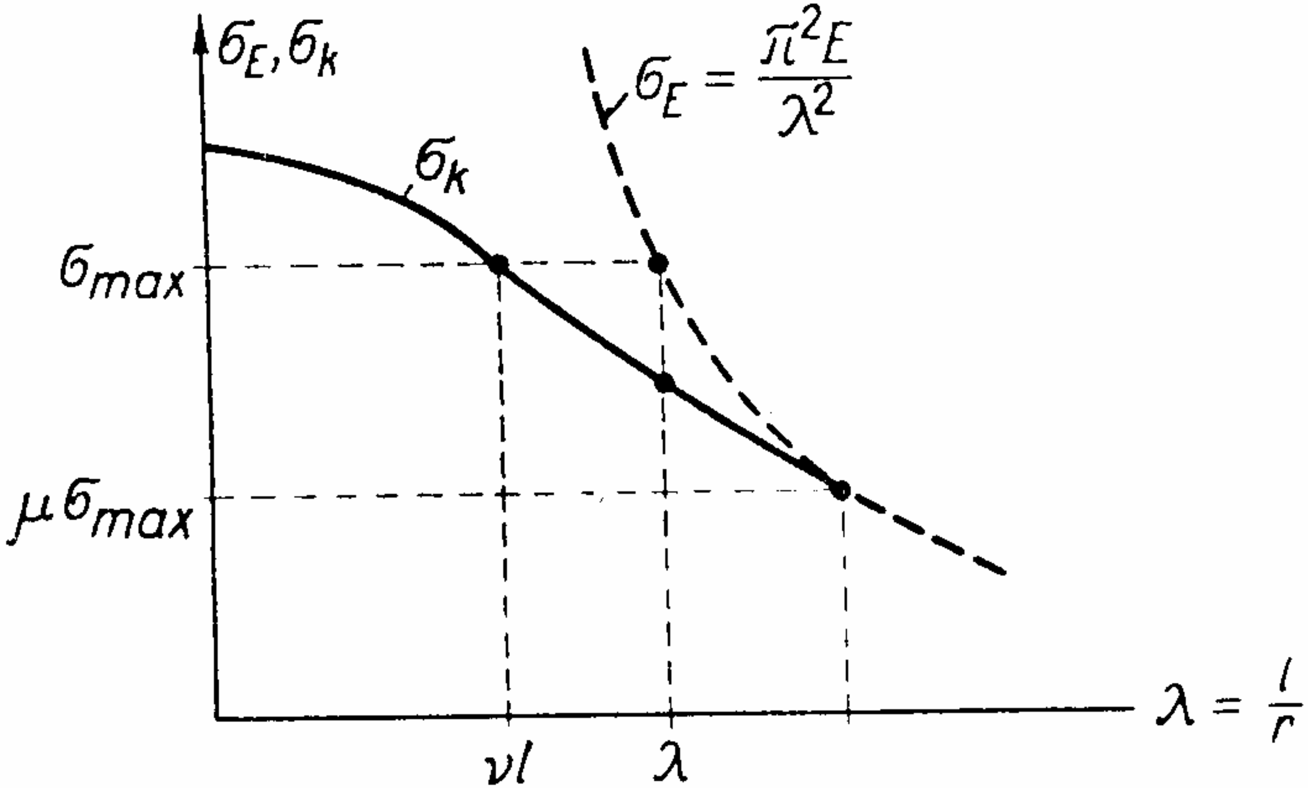
$$A = 0.5 + 0.028(\alpha+1); \qquad \text{if} \qquad -1 \leq \alpha < 5;$$

$$A = 1 - \frac{1}{2\sqrt[4]{\alpha}} \qquad \text{if} \qquad 5 \leq \alpha;$$

$$B = 0.5-0.125\sqrt{\alpha+1}; \qquad \text{if} \qquad -1 \leq \alpha < 0;$$

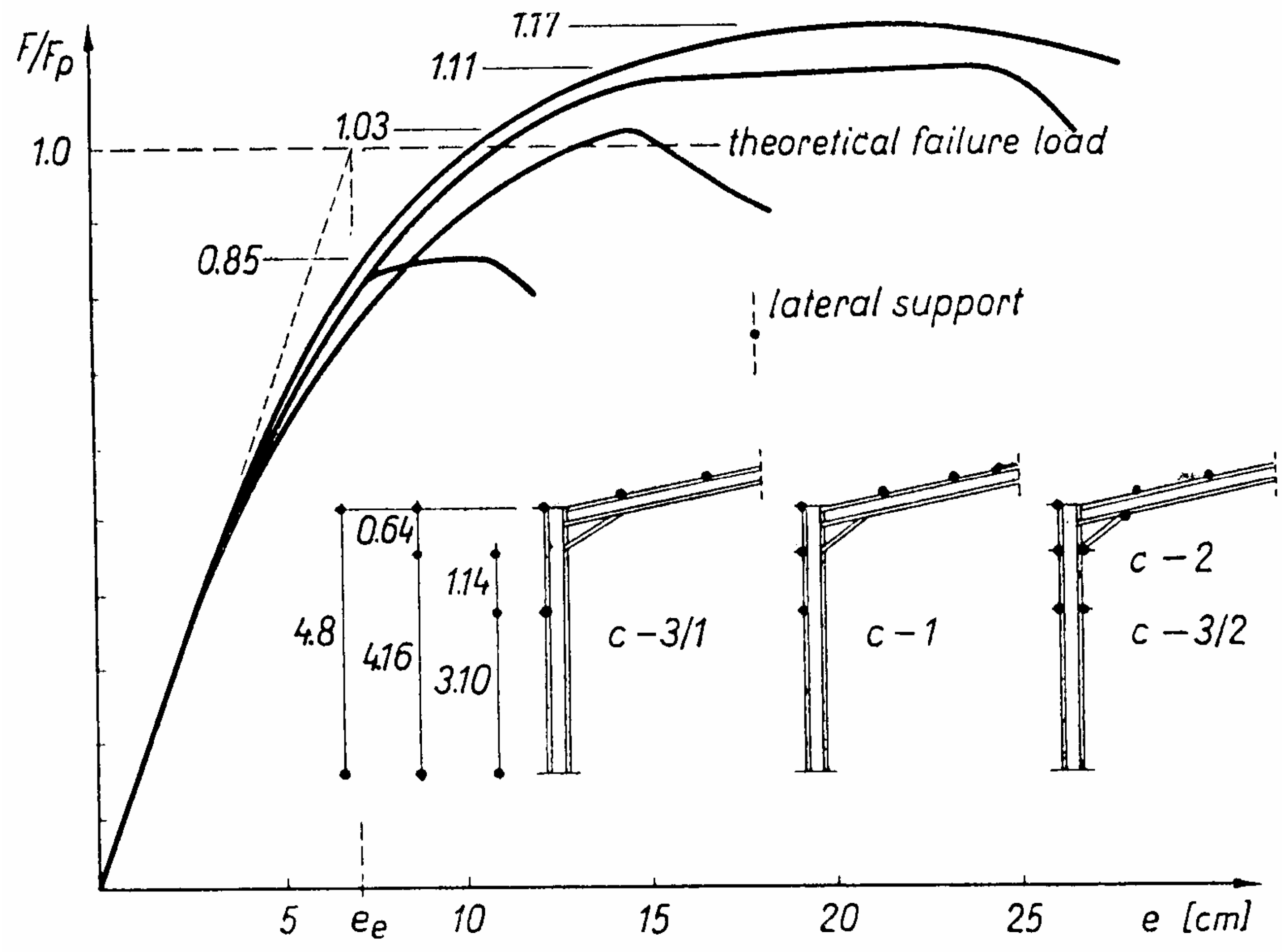
$$B = \frac{1}{2\sqrt[3]{\alpha+24}} \qquad \text{if} \qquad 0 \leq \alpha;$$

$$C = 1-A-B.$$


σE, σk
σE = π²E / λ²
σk
σmax
μσmax
νl
λ
λ = l / r

## *5.3.1.4 Test Results [Halász, Iványi, 1979]*

### *(a) Effect of lateral buckling of beam-columns*




### *(b) Effects of change in geometry*

→ Rankine-Merchant formula: $F_f = \dfrac{F_p}{0,9 + \dfrac{F_p}{F_{cr}}}$

## 5.3.2 Draft of Hungarian Specifications for Plastic Design

[Halász, Iványi, 1979]

$$EJ_\omega \theta'' - (GJ_T - Nr_p^2)\,\theta - Mu = 0$$

$$EJ_y u'' - M\theta + Nu = 0,$$

with boundary conditions $\theta = \theta'' = u = u'' = 0$ at both ends, where $EJ_y$ and $GJ_T$ denote flexural rigidity (due to the weak axis) and torsional rigidity, respectively, $J_\omega$ the warping modulus $J_\omega = \frac{D^2}{4} J_y$, $D$ being the distance between the centers of the flanges, $r_p$ the polar radius of gyration, $u$ and $\theta$ the lateral displacement and rotation of cross sections, respectively.

$$\left(\frac{M_{cr}}{M_E}\right)^2 - \left(1 - \frac{N_{cr}}{N_E}\right)\left(1 - \frac{N_{cr}}{N_\omega}\right) = 0,$$

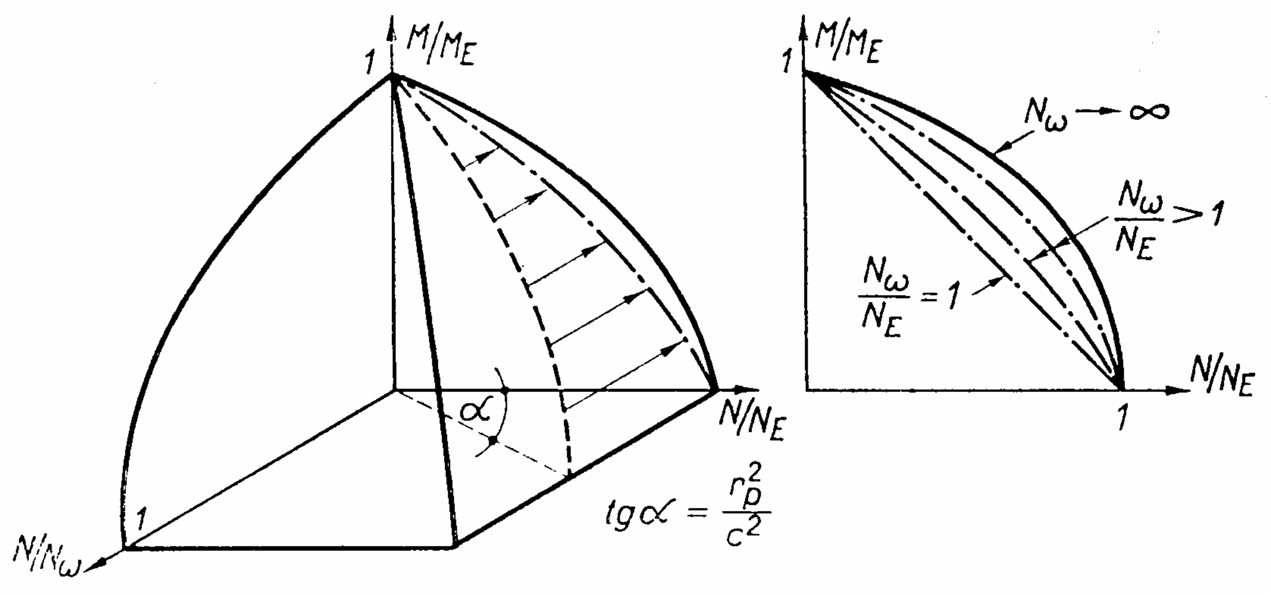
where

$N_E = \frac{\pi^2 E J_y}{l^2}$, Euler-load of a centrally compressed column due to the weak axis buckling;

$N_\omega = N_E \frac{c^2}{r_p^2}$, critical load of a centrally compressed column due to torsional buckling,

$M_E = N_E \cdot c$, elastic critical moment of the column under uniform moment;

$$c = \sqrt{\frac{GJ_T}{N_E} + \frac{J_\omega}{J_y}} = \sqrt{\frac{GJ_T}{N_E} + \frac{D^2}{4}}\ .$$




$$\sigma_{cr} = \frac{N_{cr}}{A} + \frac{M_{cr}}{W_x} = \frac{N_{cr}}{A}\left(1+\frac{d}{m}\right),$$

$$\sigma_{cr} = \frac{N_{cr}}{A}\left(1+\frac{d}{m}\right) = \frac{\pi^2 E}{\lambda_i^2}$$

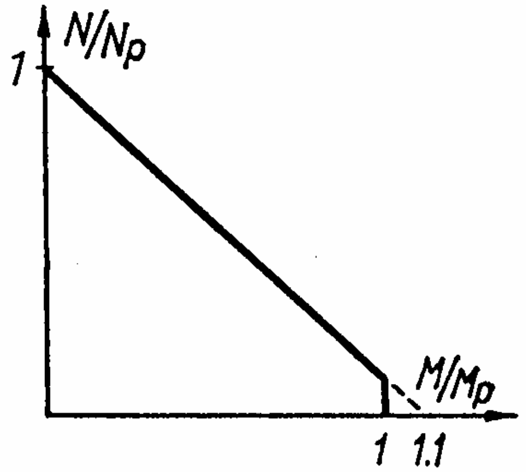
**This leads (with approximation $m \sim D/2$) to**

$$\lambda_i = \lambda_y \sqrt{\frac{s_1 + \sqrt{1+s_2^2}}{e\left(1+\frac{m}{d}\right)}}$$

**where**

$$\lambda_y = \frac{l}{r_y}; \quad H = \frac{G J_T A}{\pi^2 E W_x^2} \cong 0{,}04 \frac{K}{Am^2};$$

$$e = \sqrt{1 + H\lambda_y^2}; \quad s_{1,2} = \frac{m^2 e \pm r_p^2}{2m.d.e}.$$




$$\frac{N}{N_p} + \frac{M}{1{,}1\,M_p} \le 1$$

$$\frac{M}{M_p} \le 1$$

$$\varphi(\lambda_i) = \sigma_K/\sigma_Y: \quad \frac{N}{N_p} + \frac{M}{1{,}1\,M_p} \le \varphi(\lambda_i)\,.$$

$$\psi = \frac{1}{1 - \dfrac{N}{N_{E,x}}} \qquad \frac{N}{N_p} + \frac{\psi\mu\,M_{\max}}{1{,}1\,M_p} \le \varphi(\lambda_i)\,.$$

$$(EJ_\omega + EJ_y f^2)\theta'' - [GJ_T - Nr_f^2 + 2\,Mf]\,\theta = 0,$$

with end conditions $\theta = \theta'' = 0$; where $r_f^2 = r_p^2 + f^2$

$$\frac{M_{cr}}{\widetilde{M}_E} + \frac{N_{cr}}{\widetilde{N}_\omega} = 1$$

$$-\widetilde{N}_\omega = N_E \frac{\dfrac{J_\omega}{J_y} + \dfrac{G\,J_T}{N_E} + f^2}{r_f^2} \;,$$

$$-\widetilde{M}_E = \widetilde{N}_\omega \cdot \frac{r_f^2}{2f} \;.$$

$$\lambda_i = \sqrt{\frac{2}{1+\dfrac{m}{d}} \cdot \frac{\dfrac{r_f^2}{D\cdot d} + \dfrac{2f}{D}}{1+\left(\dfrac{2f}{D}\right)^2 + H\lambda_y^2}}$$

$$\frac{N}{N_p} + \frac{\psi\mu\, M_{\max}}{1{,}1\, M_p} \leq \varphi(\lambda_i)$$

$$\alpha = \frac{1}{1+\left(\dfrac{2f}{D}\right)^2}\left[H\lambda_y^2 - \frac{N}{N_p}\frac{r_f^2}{m^2}\left(\frac{\lambda_y}{100}\right)^2\right] .$$